# Computation of Solutions to Certain Nonlinear Systems of Fuzzy Relation Inequations

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**Abstract.** Although fuzzy relation equations and inequations have a broad field of application, it is common that they have no solutions or have only the trivial solution. Therefore, it is desirable to study new types of fuzzy relation inequations similar to the well-studied ones and with nontrivial solutions. This paper studies fuzzy relation inequations that include the degree of subsethood and the degree of equality of fuzzy sets. We provide formulae for determining the greatest solutions to systems of such fuzzy relation inequations. We provide alternative ways to compute these solutions when we cannot run the methods based on these formulae.

**Keywords:** Fuzzy relation equation · Fuzzy relation · Degree of equality

## 1 Introduction

In many scientific and technological areas, such as image processing, fuzzy control and data processing, drawing conclusions based on vague and imprecise data is required. As a primary mechanism to formalize the connection between such fuzzy data sets, fuzzy relation equations (FREs) and fuzzy relation inequations (FRIs) have been widely studied.

Linear systems of fuzzy relation equations and inequations, i.e., systems in which the unknown fuzzy relation appears only on one side of the sign $=$ or $\leqslant$, were introduced and studied in [13,14]. Linear systems were firstly considered over the Gödel structure, and later, the same systems were studied over broader sets of truth values, such as complete residuated lattices [1–3,7,18]. Afterward, nonlinear systems of fuzzy relation inequations have been examined. Among others, the so-called homogeneous and heterogeneous weakly linear systems have been introduced and studied by Ignjatović et al. [8–10] over a complete residuated

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lattice. These are the systems where an unknown fuzzy relation appears on both sides of the sign = or $\leqslant$. Also, the authors have provided procedures for determining the greatest solutions for such systems. Although these systems have applications in many areas, including concurrency theory and social network analysis, it is common that only the trivial solution exists for such systems. Accordingly, it is preferable to modify the criteria given in [9].

In [11,15], Stanimirović et al. have introduced approximate bisimulations for fuzzy automata over a complete Heyting algebra. Following this idea, Micić et al. have introduced in [12] approximate regular and approximate structural fuzzy relations for fuzzy social networks defined also over a complete Heyting algebra. These notions are generalizations of bisimulations and regular and structural fuzzy relations, and are defined as solutions to certain generalizations of weakly linear systems of FRIs. However, the observed underlying structure of truth values is a complete Heyting algebras, a special type of complete residuated lattice with an idempotent multiplication. In this paper, we study systems of FRIs introduced in [11,12,15], but we study them over an arbitrary complete residuated lattice.

Our results are the following: We show that the set of all solutions to these systems of FRIs form a complete lattice, and therefore, there exists the greatest solution for every such system. We study further properties of such systems. We give a procedure for computing this greatest solution. Unfortunately, this procedure suffers the same shortcoming as the ones developed in [8–10]. That is, it may not finish in a finite number of steps for every complete residuated lattice. Thus, we propose two alternatives in such cases. First, we show that fuzzy relations generated by this procedure converge to the solution of these systems in the case when a complete residuated lattice is a BL-algebra defined over the $[0, 1]$ interval. And second, we show that we can compute the greatest crisp solution to such systems. In this case, a procedure always terminates in a finite number of steps. However, such solutions are smaller than or equal to the solutions obtained by the procedure for computing the greatest solutions.

## 2 Preliminaries

Since we study FRIs over complete residuated lattices, we emphasize the basic characteristics of this structure. An algebra $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ where:

1) $(L, \wedge, \vee, 0, 1)$ is a lattice bounded by 0 and 1;
2) $(L, \otimes, 1)$ is a commutative monoid in which 1 as neutral element for $\otimes$;
3) operations $\otimes$ and $\to$ satisfy condition:

$$x \otimes y \leqslant z \;\text{ iff }\; x \leqslant y \to z, \qquad \text{for each } x, y, z \in L. \tag{1}$$

is called a *residuated lattice*. In addition, (1) is often called the *adjunction property*, while we say that $\otimes$ and $\to$ form the *adjoint pair*. Residuated lattice in which $(L, \wedge, \vee)$ is a complete lattice, is called a *complete residuated lattice*.

For presenting conjunction and implication, operations $\otimes$ (called *multiplication*) and $\to$ (called *residuum*) are used, and for the general and existential quantifier the infimum ($\bigwedge$) and supremum ($\bigvee$) are used, respectively. The equivalence of truth values is presented by operation *biresiduum* (or *biimplication*, denoted by $\leftrightarrow$ and defined by:

$$x \leftrightarrow y = (x \to y) \wedge (y \to x), \quad x, y \in L.$$

The well-known properties of operators $\otimes$ and $\to$ are: operator $\otimes$ is non-decreasing with respect to $\leqslant$ in both arguments, while operator $\to$ is non-decreasing with respect to $\leqslant$ in the second argument, whereas it is non-increasing in the first argument. The most studied and applied structures of truth values are the *product* (*Goguen*) *structure*, the *Gödel structure* and the *Łukasiewicz structure*. For their definitions, as well for other properties of complete residuated lattices, we refer to [4,5].

If in a residuated lattice for every $x, y \in L$, the following holds:

1. $(x \to y) \vee (y \to x) = 1$ (*prelinearity*);
2. $x \otimes (x \to y) = x \wedge y$ (*divisibility*);

then it is called a *BL-algebra*. The algebra $([0,1], \min, \max, \otimes, \to, 0, 1)$, where operation $\to$ is given by following formula:

$$x \to y = \bigvee \{z \in L \mid x \otimes z \leqslant y\},$$

is a BL-algebra if and only if operation $\otimes$ is a continuous t-norm (cf. [5, Theorem 1.40]). Specially, the Goguen (product) structure is a BL-algebra, as well as Gödel and Łukasiewicz.

In the sequel, we assume that $L$ is a support set of some complete residuated lattice $\mathcal{L}$. For a nonempty set $A$, a mapping from $A$ into $L$ is called a *fuzzy subset* of a set $A$ *over* $\mathcal{L}$, or just a *fuzzy subset* of $A$. The inclusion (ordering) and the equality of fuzzy sets are defined coordinate-wise (cf. [4,5]). The set of all fuzzy subsets of $A$ is denoted with $L^A$. For two fuzzy sets $\alpha_1, \alpha_2 \in L^A$ the meet and the join of $\alpha_1$ and $\alpha_2$ are defined as fuzzy subsets of $A$ in the following way:

$$(\alpha_1 \wedge \alpha_2)(a) = \alpha_1(a) \wedge \alpha_2(a), \quad \text{and} \quad (\alpha_1 \vee \alpha_2)(a) = \alpha_1(a) \vee \alpha_2(a),$$

for every $a \in A$. A fuzzy subset of $\alpha \in L^A$, such that rang of $\alpha$ is $\{0,1\} \subseteq L$, is called a crisp subset of the set $A$. Further, with $2^A$ we denote the set $\{\alpha \mid \alpha \subseteq A\}$. For a fuzzy subset $\alpha \in L^A$, the crisp part of $\alpha$ is the crisp set $\alpha^c \in 2^A$ defined by $\alpha^c(a) = 1$, if $\alpha(a) = 1$, and $\alpha^c(a) = 0$, if $\alpha(a) \neq 1$, for every $a \in A$. In other words, we have that $\alpha^c = \{a \in A | \alpha(a) = 1\}$.

A *fuzzy relation* on a set $A$ is any fuzzy subset of $A \times A$, or in other words, it is any function from $A \times A$ to $L$. The set of all fuzzy relations on $A$ is denoted with $L^{A \times A}$. A crisp relation is a fuzzy relation that takes values only in the set $\{0,1\}$. The set of all crisp relations on a set $A$ is denoted by $2^{A \times A}$. The universal relation on a set $A$, denoted by $u_A$, is defined as $u_A(a_1, a_2) = 1$ for all $a_1, a_2 \in A$.

*The composition* of fuzzy relations $\varphi, \phi \in L^{A\times A}$ is a fuzzy relation $\varphi \circ \phi \in L^{A\times A}$ defined by:

$$(\varphi \circ \phi)(a_1, a_2) = \bigvee_{a_3 \in A} \varphi(a_1, a_3) \otimes \phi(a_3, a_2), \quad \text{for every } a_1, a_2 \in A. \tag{2}$$

Note that the composition of fuzzy relations is an associative operation on a set $L^{A\times A}$. For $x \in L$ and $\varphi \in L^{A\times A}$, we define fuzzy relations $x \otimes \varphi \in L^{A\times A}$ and $x \to \varphi \in L^{A\times A}$ as

$$(x \otimes \varphi)(a_1, a_2) = x \otimes \varphi(a_1, a_2), \quad \text{and} \quad (x \to \varphi)(a_1, a_2) = x \to \varphi(a_1, a_2), \tag{3}$$

for every $a_1, a_2 \in A$. We assume that $\circ$ has a higher precedence than $\otimes$ and $\to$ defined by (3). For $\varphi, \phi_1, \phi_2 \in L^{A\times A}$, family $\varphi_i \in L^{A\times A} (i \in I)$ and $x \in L$, the following holds:

$$\phi_1 \leqslant \phi_2 \quad \text{implies} \quad \varphi \circ \phi_1 \leqslant \varphi \circ \phi_2 \text{ and } \phi_1 \circ \varphi \leqslant \phi_2 \circ \varphi, \tag{4}$$

$$\phi \circ \left( \bigvee_{i\in I} \varphi_i \right) = \bigvee_{i \in I} (\phi \circ \varphi_i), \; \left( \bigvee_{i\in I} \varphi_i \right) \circ \phi = \bigvee_{i \in I} (\varphi_i \circ \phi). \tag{5}$$

For $\alpha_1, \alpha_2 \in L^A$, the *degree of subsethood of* $\alpha_1$ *in* $\alpha_2$, denoted with $S(\alpha_1, \alpha_2) \in L$, is defined by:

$$S(\alpha_1, \alpha_2) = \bigwedge_{a \in A} \alpha_1(a) \to \alpha_2(a), \tag{6}$$

while the *degree of equality of* $\alpha_1$ *and* $\alpha_2$, denoted with $E(\alpha_1, \alpha_2) \in L$, is defined as:

$$E(\alpha_1, \alpha_2) = \bigwedge_{a \in A} \alpha_1(a) \leftrightarrow \alpha_2(a). \tag{7}$$

Intuitively, $S(\alpha_1, \alpha_2)$ can be understood as a truth degree of the statement that if some element of $A$ belongs to $\alpha_1$, then this element belongs to $\alpha_2$. Also, $E(\alpha_1, \alpha_2)$ can be understood as a truth degree of the statement that an element of $A$ belongs to $\alpha_1$ if and only if it belongs to $\alpha_2$. (see [5] for more details). It can easily be proved that $E(\alpha_1, \alpha_2) = S(\alpha_1, \alpha_2) \wedge S(\alpha_2, \alpha_1)$.

It should be noted that the degree of subsethood is a kind of residuation operation that assigns a scalar to a pair of fuzzy sets. That operation is the residual of the operation of multiplying the fuzzy set by a scalar. Thus, the fuzzy relation inequations considered in this paper can be connected with fuzzy relation equations defined using residuals known in the literature.

Let $\varphi, \psi \in L^{A\times A}$ be fuzzy relations. Then we define the *right residual* $\varphi \backslash \psi \in L^{A\times A}$ of $\psi$ *by* $\varphi$ and the *left residual* $\psi / \varphi \in L^{A\times A}$ *of* $\psi$ *by* $\varphi$, respectively, by the following formulae:

$$(\varphi \backslash \psi)(a_1, a_2) = S(\varphi a_1, \psi a_2), \quad \text{for every } a_1, a_2 \in A$$
$$(\psi / \varphi)(a_1, a_2) = S(a_2 \varphi, a_1 \psi), \quad \text{for every } a_1, a_2 \in A.$$

It can be shown that the right and left residuals satisfy the following two adjunction properties:

$$\varphi \circ \chi \leqslant \psi \quad \text{iff} \quad \chi \leqslant \varphi \backslash \psi, \tag{8}$$

$$\chi \circ \varphi \leqslant \psi \quad \text{iff} \quad \chi \leqslant \psi / \varphi, \tag{9}$$

where $\chi \in L^{A\times A}$ an arbitrary fuzzy relation on $A$. For other properties of residuals of fuzzy relations we address to [8–10]. In addition, we define crisp relations $\varphi \,\overline{\backslash}\, \psi \in 2^{A\times A}$ and $\psi \,\overline{/}\, \varphi \in 2^{A\times A}$, called the *Boolean right residual of* $\psi$ *by* $\varphi$ and the *Boolean left residual of* $\psi$ *by* $\varphi$), respectively, in the following way:

$$\varphi \,\overline{\backslash}\, \psi = (\varphi\backslash\psi)^c = \{(a_1, a_2) \in A \times A | \varphi a_1 \leqslant \psi a_2\}, \qquad \text{for every } a_1, a_2 \in A,$$
$$\psi \,\overline{/}\, \varphi = (\psi/\varphi)^c = \{(a_1, a_2) \in A \times A | a_2\varphi \leqslant a_1\psi\}, \qquad \text{for every } a_1, a_2 \in A.$$

Again, if $X \in 2^{A\times A}$, then the following two adjunction properties hold:

$$\varphi \circ X \leqslant \psi \quad \text{iff} \quad X \leqslant \varphi \,\overline{\backslash}\, \psi, \tag{10}$$

$$X \circ \varphi \leqslant \psi \quad \text{iff} \quad X \leqslant \psi \,\overline{/}\, \varphi, \tag{11}$$

For properties on Boolean right and Boolean left residuals in the context of matrices over additively idempotent semirings, we refer to [6,16].

## 3 Certain Types of Fuzzy Relation Inequations and Their Solutions

For a given nonempty set $A$, a family $\{\varrho_i\}_{i\in I}$ of fuzzy relations on $A$, and a scalar $\lambda \in L$ from a complete residuated lattice $\mathcal{L}$, consider the following systems of fuzzy relation inequations:

$$S(\varphi \circ \varrho_i, \varrho_i \circ \varphi) \geqslant \lambda \quad \text{for every } i \in I, \tag{12}$$

$$S(\varrho_i \circ \varphi, \varphi \circ \varrho_i) \geqslant \lambda \quad \text{for every } i \in I, \tag{13}$$

$$E(\varphi \circ \varrho_i, \varrho_i \circ \varphi) \geqslant \lambda \quad \text{for every } i \in I, \tag{14}$$

where $\varphi \in L^{A\times A}$ is an unknown fuzzy relation on $A$. Note that (14) is equivalent to the conjunction of (12) and (13). According to the definition (6) and the adjunction property, we conclude that a fuzzy relation $\varphi$ is a solution to (12) if and only if:

$$\lambda \otimes \varphi \circ \varrho_i \leqslant \varrho_i \circ \varphi \quad \text{for all } i \in I, \tag{15}$$

and similarly, $\varphi$ is a solution to (13) if and only if:

$$\lambda \otimes \varrho_i \circ \varphi \leqslant \varphi \circ \varrho_i \quad \text{for all } i \in I. \tag{16}$$

**Theorem 1.** *The sets of all solutions to* (12), (13) *and* (14) *that are subsets of* $\varphi_0 \in L^{A\times A}$ *is a complete lattice. Accordingly, there exists the greatest solution* $\varphi$ *to* (12), (13) *and* (14) *such that* $\varphi \leqslant \varphi_0$.

*Proof.* We prove that this statement holds only for (14), since the case (12) and (13) can be shown in a similar way. Note that the empty relation is a solution to the system (14), for every $\lambda \in L$, and hence the set of all solutions to (14) has at least one element. With $\{\varphi_j\}_{j\in J}$ we label the family of all solutions. Let $\psi = \bigvee_{j\in J} \varphi_j$. Using (5) we show that $\psi$ is also a solution to the system (14), and hence $\{\varphi_j\}_{j\in J}$ is a complete lattice such that $\bigvee_{j\in J} \varphi_j$ is its greatest element.

**Lemma 1.** *Let $\{\varrho_i\}_{i\in I}$ a family of fuzzy relations on a set $A$, and let $\varphi_1, \varphi_2 \in L^{A\times A}$ such that $\varphi_1$ is the greatest solution to (14) when $\lambda = \lambda_1$, and $\varphi_2$ is the greatest solutions to (14) when $\lambda = \lambda_2$, for some $\lambda_1, \lambda_2 \in L$ such that $\lambda_1 \leqslant \lambda_2$. Then $\varphi_2 \leqslant \varphi_1$.*

*Proof.* Denote with $R_{\lambda_1}$ and $R_{\lambda_2}$ the set of all fuzzy relations that are solutions to (14) when $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Evidently every solution from $R_{\lambda_2}$ belongs also to $R_{\lambda_1}$. Therefore, the greatest solution from $R_{\lambda_2}$ is contained in $R_{\lambda_1}$. This implies $\varphi_2 \leqslant \varphi_1$.

**Lemma 2.** *Let $\{\varrho_i\}_{i\in I}$ a family of fuzzy relations on a set $A$. Let $\lambda_1, \lambda_2 \in L$ be two values, such that $\lambda_1 < \lambda_2$, and the greatest solution to (14) when $\lambda = \lambda_1$ is the same as the greatest solution to (14) when $\lambda = \lambda_2$. Then the greatest solution to (14) for every $\lambda \in L$ such that $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, is the same fuzzy relation.*

*Proof.* Denote with $\varphi_1$ (resp. $\varphi_2$) the greatest solution to (14) when $\lambda = \lambda_1$ (resp. $\lambda = \lambda_2$). According to the previous lemma, $\lambda_1 \leqslant \lambda \leqslant \lambda_2$ implies $\varphi_1 \leqslant \varphi_2 \leqslant \varphi_1$, and hence $\varphi_1 = \varphi_2$.

Theorem 1 states that, for all systems of the form (12), (13) and (14), there exists the greatest solution, but it does not provide a way to compute it. Here, we provide a method for determining these greatest solutions. Precisely, we propose a function for finding the greatest solution to (14) contained in a given fuzzy relation. The greatest solutions to (12) and (13) can be obtained in the similar way. A variant of the following result is proven in [17], so we omit its proof.

For a given family $\{\varrho_i\}_{i\in I}$ and an element $\lambda \in L$, define two functions $\mathcal{Q}_1(\lambda), \mathcal{Q}_2(\lambda) : L^{A\times A} \to L^{A\times A}$, for every $\varphi \in L^{A\times A}$, as:

$$\mathcal{Q}_1(\lambda)(\varphi) = \bigwedge_{i\in I} (\lambda \to \varrho_i \circ \varphi)/\varrho_i, \tag{17}$$

$$\mathcal{Q}_2(\lambda)(\varphi) = \bigwedge_{i\in I} \varrho_i \backslash (\lambda \to \varphi \circ \varrho_i). \tag{18}$$

**Theorem 2.** *Let $\varphi_0 \in L^{A\times A}$ be a fuzzy relation on $A$, and let $\lambda \in L$. Consider a procedure for computing the array $\{\varphi_n\}_{n\in\mathbb{N}_0}$ of fuzzy relations on $A$ by:*

$$\varphi_{n+1} = \varphi_n \wedge \mathcal{Q}_1(\lambda)(\varphi_n) \wedge \mathcal{Q}_2(\lambda)(\varphi_n), \tag{19}$$

*for every $n \in \mathbb{N}_0$. Then the following holds:*

*a) $\varphi_k$ is the greatest solution to (14) contained in $\varphi_0$ if and only if $\varphi_k = \varphi_{k+1}$.*
*b) If $\mathcal{L}(\{\varrho\}_{i\in I}, \lambda)$ is a finite subalgebra of $\mathcal{L}$, procedure (19) produces the greatest solution to (14), contained in $\varphi_0$, in a finite number of steps.*

According to the previous result we obtain a function for computing the greatest solution to (14) contained in a fuzzy relation $\varphi_0$, for a given $\lambda \in L$ and $\varphi_0 \in L^{A\times A}$, formalized by Function 1.

**Function 1.** $\mathsf{ComputeTheGreatestSolution}(\{\varphi_n\}_{n\in\mathbb{N}_0}, \varphi_0, \lambda)$

Input: *a family* $\{\varrho_i\}_{i\in I}$*, a fuzzy relation* $\varphi_0 \in L^{A\times A}$ *and an element* $\lambda \in L$.

Output: *a fuzzy relation that is the greatest solution to system* (14) *contained in* $\varphi_0$.

*1.* $\phi = \varphi_0$;
*2. do:*
*3.* $\varphi = \phi$;
*4.* $\phi = \phi \wedge \mathcal{Q}_1(\lambda)(\phi) \wedge \mathcal{Q}_2(\lambda)(\phi)$;
*5. while*$(\phi \neq \varphi)$;
*6. return* $\phi$;

If the previous function is unable to compute the output in a finite number of steps, we discuss alternative ways to obtain this fuzzy relations. Precisely, the exact solution to (14), contained in $\varphi_0$ is obtained when the above function finishes after a finite number of steps. If that is not the case, then when $\mathcal{L}$ is a BL-algebra, then the array of fuzzy relations computed using Function 1 is convergent, and the limit value of this array is the sought solution. According to this fact, in the case when Function 1 does not finish after limited number of steps, we can use Theorem 3 to compute the greatest solution to system (14) contained in a given fuzzy relation.

**Theorem 3.** *Let* $\varphi_0 \in L^{A\times A}$ *be a fuzzy relation on* $A$*, and let* $\lambda \in L$*, where* $\mathcal{L}$ *is a BL-algebra over the interval* $[0,1]$*. Then the sequence* $\{\varphi_n\}_{n\in\mathbb{N}}$ *of fuzzy relations on* $A$*, defined by* (19)*, is convergent. If we denote* $\lim_{n\to\infty}\varphi_n = \widetilde{\varphi}$*, then* $\widetilde{\varphi}$ *is the greatest solution to* (14) *contained in* $\varphi_0$.

*Proof.* By the construction of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$, we can easily conclude that it is monotonic decreasing. In addition, since every element of this sequence is greater than the zero relation and less than the universal relation, we conclude that it is also bounded. The array $\{\varphi_n\}_{n\in\mathbb{N}}$ is convergent because it is monotonic and bounded. Denote with $\widetilde{\varphi}$ a fuzzy relation that is the limit value of this array.

Since the array $\{\varphi_n\}_{n\in\mathbb{N}}$ is non-increasing, we have that $\widetilde{\varphi} \leqslant \varphi_n$ holds for every $n \in \mathbb{N}$, and thus, $\widetilde{\varphi}$ is contained in $\varphi_0$. For proving that $\widetilde{\varphi}$ is a solution to (14), we need to prove that $\widetilde{\varphi}$ satisfies (15) and (16). From the definition of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$, we conclude that for every $n \in \mathbb{N}$ it holds:

$$\varphi_{n+1} \leqslant (\lambda \to \varrho_i \circ \varphi_n)/\varrho_i \quad \text{and} \quad \varphi_{n+1} \leqslant \varrho_i \backslash (\lambda \to \varphi_n \circ \varrho_i), \qquad \text{for every } i \in I.$$

According to the definitions of right and left residuals, the previous inequations are equivalent to:

$$\varphi_{n+1} \circ \varrho_i \leqslant \lambda \to \varrho_i \circ \varphi_n \quad \text{and} \quad \varrho_i \circ \varphi_{n+1} \leqslant \lambda \to \varphi_n \circ \varrho_i, \qquad \text{for every } i \in I,$$

which are, by the adjunction property, equivalent to:

$$\lambda \otimes \varphi_{n+1} \circ \varrho_i \leqslant \varrho_i \circ \varphi_n \quad \text{and} \quad \lambda \otimes \varrho_i \circ \varphi_{n+1} \leqslant \varphi_n \circ \varrho_i, \qquad \text{for every } i \in I.$$

According to the fact that the t-norm $\otimes$ is continuous in BL-algebras, from the previous inequalities we obtain that for every $i \in I$ the following inequality holds:

$$\lambda \otimes \widetilde{\varphi} \circ \varrho_i = \lambda \otimes \left(\lim_{n\to\infty} \varphi_n\right) \circ \varrho_i = \lim_{n\to\infty} (\lambda \otimes \varphi_n \circ \varrho_i)$$
$$\leqslant \lim_{n\to\infty} (\varrho_i \circ \varphi_{n-1}) = \varrho_i \circ \left(\lim_{n\to\infty} \varphi_{n-1}\right) = \varrho_i \circ \widetilde{\varphi}.$$

Thus, $\widetilde{\varphi}$ is solution to (12). Analogously, we prove that $\widetilde{\varphi}$ is solution to (13), which means that $\widetilde{\varphi}$ is solution to (14).

We now show that an arbitrary solution to (14), contained in $\varphi_0$, is less than every fuzzy relation $\varphi_n$ ($n \in \mathbb{N}$), obtained by procedure (19). Let $\psi$ be a solution to (14), contained in $\varphi_0$. Evidently, $\psi \leqslant \varphi_0$ Suppose that $\psi \leqslant \varphi_n$, for some $n \in \mathbb{N}^0$. Then for every $i \in I$ we have $\lambda \otimes \psi \circ \varrho_i \leqslant \varrho_i \circ \psi \leqslant \varrho_i \circ \varphi_n$, which implicate $\psi \leqslant (\lambda \to \varrho_i \circ \varphi_n)/\varrho_i$, and similarly $\psi \leqslant \lambda \to \varphi_n \circ \varrho_i$ for every $i \in I$, we have:

$$\psi \leqslant \varphi_n \wedge \bigwedge_{i\in I} (\lambda \to \varrho_i \circ \varphi_n)/\varrho_i \wedge \bigwedge_{i\in I} \varrho_i \setminus (\lambda \to \varphi_n \circ \varrho_i) = \varphi_{n+1}.$$

According to the mathematical induction it follows $\psi \leqslant \varphi_n$, for every $n \in \mathbb{N}$. Hence, $\psi \leqslant \lim_{n\to\infty} \varphi_n = \widetilde{\varphi}$.

The following example shows the case of a system (14) when Function 1 doesn't terminate in a finite number of steps by putting $\lambda = 1$. On the other hand, by putting $\lambda = 0.7$, Function 1 is able to compute the greatest solution to (14).

*Example 1.* Consider the system (14) over the product structure, where $\{\varrho_i\}_{i\in I} = \{\varrho\}$, and relation $\varrho \in L^{A\times A}$ is defined with:

$$\varrho = \begin{bmatrix} 0.9 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.8 & 0 & 0.3 & 0 & 0.2 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0.4 \\ 0 & 0 & 0.8 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & 1 & 0.2 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0.1 \end{bmatrix}. \tag{20}$$

Let $\lambda = 0.7$. Then the greatest solution to (14), contained in $u_A$, outputted by Function 1, is given by:

$$\varphi = \begin{bmatrix} 1 & 1 & 1 & 1 & 50/63 & 40/63 \\ 1 & 1 & 1 & 1 & 5/7 & 4/7 \\ 1 & 1 & 1 & 1 & 50/63 & 40/63 \\ 1 & 1 & 1 & 1 & 5/7 & 4/7 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{21}$$

On the other hand, if we set $\lambda = 1$, we get that the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$, generated by formula (19), is infinite. Therefore, Function 1 cannot be used to compute the greatest solution to (14). Note that the greatest solution to (14) is equal to

$$\varphi = \lim_{n\to\infty} \varphi_n = \begin{bmatrix} 1 & 50/81 & 400/729 & 50/81 & 5/9 & 1600/6561 \\ 0 & 1 & 64/81 & 32/81 & 8/81 & 32/81 \\ 0 & 8/9 & 1 & 2/5 & 1/10 & 2/5 \\ 0 & 8/9 & 4/5 & 1 & 1/5 & 32/81 \\ 0 & 1 & 80/81 & 1 & 1 & 320/729 \\ 0 & 1 & 9/10 & 9/20 & 1/5 & 1 \end{bmatrix}.$$

As we have stated, Function 1 for computing the greatest solution to (14) does not necessary terminate in the finite number of steps. In that case, it is possible to adjust the procedure to compute the greatest crisp solution to (14) contained in a given crisp relation.

For a given family $\{\varrho_i\}_{i\in I}$ and an element $\lambda \in L$, define two functions $\mathcal{Q}_1^c(\lambda), \mathcal{Q}_2^c(\lambda) : L^{A\times A} \to L^{A\times A}$, for every $\varphi \in L^{A\times A}$, as:

$$\mathcal{Q}_1^c(\lambda)(\varphi) = \bigwedge_{i\in I} (\lambda \to \varrho_i \circ \varphi) \nearrow \varrho_i, \tag{22}$$

$$\mathcal{Q}_2^c(\lambda)(\varphi) = \bigwedge_{i\in I} \varrho_i \nwarrow (\lambda \to \varphi \circ \varrho_i). \tag{23}$$

**Theorem 4.** *Let $\varphi_0 \in 2^A$ be a crisp relation on $A$ and $\lambda \in L$. Define an array $\{\varphi_n\}_{n\in\mathbb{N}}$ of crisp relations on $A$ by:*

$$\varphi_{n+1} = \varphi_n \wedge \mathcal{Q}_1^c(\lambda)(\varphi_n) \wedge \mathcal{Q}_2^c(\lambda)(\varphi_n), \quad n \in N. \tag{24}$$

*Then:*

1. *Relation $\varphi_k$ is the greatest crisp solution to (14), such that $\varphi_k \subseteq \varphi_0$ if and only if $\varphi_k = \varphi_{k+1}$.*
2. *Procedure (24) produces the greatest crisp solution to (14) such that $\varphi_k \subseteq \varphi_0$, after finitely many iterations.*

*Proof.* The proof follows similar as the proof for Theorem 2, so it is omitted.

According to Theorem 4, we give an algorithm for computing the greatest crisp solution to (14), formalized by Function 2.

**Function 2.** ComputeTheGreatestCrispSolution$(\{\varphi_n\}_{n\in\mathbb{N}_0}, \varphi_0, \lambda)$

Input: *a family $\{\varrho_i\}_{i\in I}$, a crisp relation $\varphi_0 \in 2^A$ and an element $\lambda \in L$.*

Output: *a crisp relation that is the greatest solution contained in $\varphi_0$ to system (14).*

1. $\phi = \varphi_0$;
2. *do:*
3. $\varphi = \phi$;
4. $\phi = \phi \wedge \mathcal{Q}_1^c(\lambda)(\phi) \wedge \mathcal{Q}_2^c(\lambda)(\phi)$;
5. *while*$(\phi \neq \varphi)$;
6. *return* $\phi$.

The following example demonstrates the case when it is impossible to compute the greatest solution to (14), but it is possible to determine the greatest solution to (14) by means of Theorems 3 and 4.

*Example 2.* Consider the system (14) over the product structure, where $\{\varrho_i\}_{i\in I} = \{\varrho_1, \varrho_2\}$, where fuzzy relations $\varrho_1$ and $\varrho_2$ are given by:

$$\varrho_1 = \begin{bmatrix} 1 & 0.95 & 0.9 & 1 & 0.9 & 0.9 & 0.8 & 0.8 \\ 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 1 & 0.4 & 0 & 0 & 0.95 & 0.4 \\ 0 & 0.95 & 0 & 0 & 0.4 & 0 & 0.9 & 0.95 \\ 0.8 & 0.95 & 0.9 & 0.95 & 0.95 & 0.8 & 1 & 0.4 \\ 0.4 & 1 & 0.6 & 0 & 0.6 & 0.6 & 0.9 & 0.9 \\ 0 & 0.95 & 0.4 & 0.9 & 1 & 0.9 & 0.8 & 0.95 \\ 0 & 0.4 & 1 & 0.8 & 0.9 & 1 & 0.95 & 0 \end{bmatrix}, \tag{25}$$

$$\varrho_2 = \begin{bmatrix} 0.8 & 0.95 & 1 & 0.9 & 0.95 & 0.9 & 0 & 0.9 \\ 0.95 & 0.9 & 0 & 0.6 & 0 & 1 & 0.9 & 0 \\ 0.4 & 1 & 0.9 & 0.95 & 0.8 & 0 & 0.4 & 0.9 \\ 0.8 & 0.6 & 1 & 0.9 & 0.9 & 0.6 & 0.6 & 0.95 \\ 0.8 & 0.6 & 0 & 0.4 & 0.4 & 0 & 0.8 & 0.95 \\ 0 & 0.6 & 0.8 & 0.4 & 1 & 0 & 0.9 & 1 \\ 0.6 & 0 & 0.95 & 0.8 & 0.4 & 0 & 0.4 & 0.8 \\ 1 & 0 & 0.95 & 0.8 & 0.8 & 0 & 0 & 0.8 \end{bmatrix}. \tag{26}$$

Take $\lambda = 0.9$. Then Function 1 does not stop in a finite number of steps, thus, it is impossible to determine the greatest solution to (14) by means of this Function. But, by using Theorem 3, we conclude that the greatest solution to (14) is given by:

$$\varphi = \lim_{n\to\infty} \varphi_n = \begin{bmatrix} 1 & 1 & 0 & 40/57 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2/3 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 20/27 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

On the other hand, if we employ Function 2, it ends after three iterations, and the output of this function is a crisp relation given by:

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

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