Bisimulations for weighted finite automata over semirings

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Abstract

Simulation and bisimulation relations are powerful tools used in many areas of computer science to match moves and compare the behaviour of various computational systems, such as labelled transition systems and automata, as well as to reduce the number of states of these systems. By moving from traditional boolean systems to quantitative ones, a need arise for both simulations and bisimulations to be quantitative, to be modeled with matrices whose entries should provide some quantitative measure of the relationship between the states of the considered systems. In the present paper, we introduce several types of quantitative simulations and bisimulations for weighted finite automata over semirings. For weighted finite automata over positive semirings two tipes of simulations and two types of bisimulations are defined as solutions to particular systems of matrix inequations, while for weighted finite automata over arbitrary semirings four types of bisimulations are defined as solutions to particular systems of matrix equations. Both types of simulations are proven to ensure the containment, while all types of bisimulations are proven to ensure the equivalence of weighted finite automata. Certain general properties of simulations and bisimulations are also proved.

Keywords: Weighted finite automaton, semiring, positive semiring, containment problem, equivalence problem, simulation, bisimulation, matrix inequations, matrix equations

1 Introduction

One of the most significant general problems in theoretical computer science is the *equivalence problem* – the question of determining, for two given models of computation, whether they do the same job. Applied to wighted finite automata (WFAs for short) this is the question of whether two WFAs compute the same word function. A related problem is the *containment problem* (also called the *comparison* or *inclusion problem*). For WFAs over an ordered semiring this is the question of determining whether the function computed by one WFA is less than or equal to the function computed by the other WFA (with respect to the pointwise ordering of functions). The complexity and decidability of these decision problems depend upon the type of WFAs under consideration, i.e., it depends both on the determinism of the automata and the underlying semiring (for more information we refer to [17, 18, 33]). In cases where the containment or equivalence problem is undecidable or computationally hard, the question naturally arises as to whether it is possible to efficiently determine "something" that implies containment or equivalence. That "something" has been provided for classical transition systems and nondeterministic automata by Milner [30] and Park [31], by introducing *simulation* and *bisimulations relations*, which proved to be very powerful tools used in many areas of computer science to match moves and compare the behaviour of various systems, as well as to reduce the number of states of these systems. Roughly at the same time they have been also discovered in some areas of mathematics, e.g., in modal logic and set theory. The use of simulations and bisimulations has gained a long and rich history and their various forms have been defined and applied to different systems. They are employed today in the study of functional languages, object-oriented languages, types, datatypes, domains, databases, compiler optimizations, program analysis, verification tools, etc.

It is especially important that simulations and bisimulations specify the relationships between the states of various transition systems, which is often not the case when determining containment or equivalence between these systems. Traditionally, simulations and bisimulations have been considered as classical boolean relations, and such a traditional approach has continued even with the transition to quantitative settings. However, to achieve the full power of quantitative analysis, simulations and bisimulations should also be quantitative, they should provide some quantitative measure of the relationship between the states of the considered systems. In the case of WFAs this means that they should be weighted relations, that is, matrices with entries in the underlying semiring. Such an approach to simulations and bisimulations was used in [14], where two types of simulations and four types of bisimulations were introduced for fuzzy finite automata (FFAs for short). They were defined as fuzzy matrices that are solutions to particular systems of matrix inequations and equations, and the central place in testing solvability and solving these systems is held by the residuation in the underlying complete residuated lattice $\mathbb{L}$, which is further transferred to the matrices with entries in $\mathbb{L}$. Based on that, algorithms for testing the existence and computing the greatest simulations and bisimulations for FFAs have been designed in [15]. These algorithms are based on iterative procedures that in some cases do not finish in a finite number of steps. For such cases, modified algorithms have been developed that test the existence and compute the greatest crisp simulations and bisimulations, those represented by boolean matrices (cf. [13, 15]). However, fuzzy simulations and bisimulations have been shown to give better results in representing containment and equivalence than their crisp versions. Namely, examples of FFAs have been given which possess fuzzy simulations and bisimulations, but not crisp simulations and bisimulations. This demonstrates the great advantages of quantitative simulations and bisimulations over the classical boolean ones.

The main question we deal with in this paper is how to transfer this approach developed for FFAs to the wider class of WFAs over semirings. In other words, the question is how to define simulations and bisimulations for WFAs over semirings using appropriate systems of matrix inequations and equations. Obviously, in order to use matrix inequations, ordering of the matrices is necessary, which is achieved when the underlying semiring is ordered. Therefore, in exactly the same way as for FFAs, for WFAs over an ordered semiring two types of simulations and two types of bisimulations can be defined using systems of matrix inequations. However, simulations and bisimulations defined in this way do not necessarily accomplish their main purpose, that simulations ensure containment and bisimulations ensure equivalence. To accomplish this purpose, we need the ordering of matrices to be compatible with all possible matrix multiplications, and we show that this can be achieved if the underlying ordered semiring is positive. In other words, for WFAs over a positive semiring, using systems of matrix inequations, we define forward and backward simulations, which ensure containment, and forward and backward bisimulations, which ensure equivalence of WFAs. Let us note that quantitative simulations by matrices, defined in the same way as here, have been studied in [33] for WFAs over a special type of positive semirings in which any increasing chain has a supremum.

Another basic question we deal with here is how to define bisimulations for WFAs over semirings which are not positive. Among the four types of bisimulations defined in [14], two types, the so-called forward-backward and backward-forward bisimulations, are defined using systems of matrix equations. Using systems of matrix equations of the same form, we also define forward-backward and backward-forward bisimulations for WFAs over an arbitrary semiring, and prove that they ensure equivalence of WFAs. It should be noted that backward-forward bisimulations have already appeared under different names in the literature on WFAs, see [4–6, 10, 21, 22, 29, 32] (for more information see also[19]). However, it turned out that one or two types of bisimulations are sometimes not enough. Namely, for each of the four types of bisimulations defined in [14] by systems of matrix inequations and equations, an example was found in [15] which demonstrates that there is a bisimulation of that type, but there are no bisimulations of any of the remaining three types. This shows us that it is important to have as many different types of bisimulations as possible, and in this particular case the question naturally arises whether some other types of bisimulations can be defined for WFAs over arbitrary semirings as an alternative to forward and backward bisimulations. We give a positive answer to that question by introducing two completely new systems of matrix equations with two unknowns and proving that the solutions to those systems ensure the equivalence of WFAs over an arbitrary semiring. We also prove that for WFAs over a positive semiring, under certain conditions, solutions to those systems become forward and backward bisimulations, and vice versa.

An extremely important issue is testing the existence and computing simulations and bisimulations. However, this issue is highly dependent on the underlying structure of weights and requires separate research for different weight structures. For example, the methods for solving the systems of matrix inequations and equations discussed in this paper are completely different when working with matrices over the field of real numbers than when working with fuzzy matrices or max-plus matrices. For this reason, we do not deal with that question here, but it is left for subsequent researches in which we will deal with that question separately for WFAs over the field of real numbers and WFAs over the max-plus semiring.

It is worth mentioning that simulations and bisimulations for max-plus automata were studied in [28] using the approach developed in [19] for WFAs over additively idempotent semirings. However, this approach is not completely quantitative because it is based on the search for solutions to systems of matrix inequations and equations in the set of boolean matrices, which leads to the considered simulations and bisimulations being classical bivalent relations. In a few papers, one can also encounter concepts related to bisimulations discussed in the context of WFAs over the field of real numbers. A linear weighted bisimulation was introduced in [9], and it was also discussed in [2], whereas a bisimulation seminorm was introduced in [2]. Both of them are defined on linear weighted automata, which are not the same as WFAs defined in this paper, but they are crisp-deterministic WFAs obtained from linear representations of ordinary WFAs, called in [12, 26] Nerode automata of these WFAs. Viewed in this way, linear weighted bisimulations are nothing but congruences (of crisp-deterministic WFAs) on Nerode automata (cf. [26]). It should also be said that linear weighted bisimulations and bisimulation seminorms are defined on a single automaton, they do not relate two different automata. Besides, none of the mentioned models of bisimulation represents a quantitative bisimulation, in the sense of a matrix that determines certain degrees of relationship between states of WFAs. Quantitative simulations have been discussed only in [33], for max-plus automata and WFAs over the semiring of nonnegative real numbers. Quantitative simulations for max-plus automata have been studied using a game-theoretical approach, while for WFAs over the semiring of nonnegative reals, an algorithm was provided for testing the existence of quantitative simulations and their computing, which is based on reduction to a linear programming problem and its solution by the simplex method.

The paper is organized as follows. After this introductory section, in Section 2 we give the definitions of semirings, ordered and positive semirings, and present the most important examples of such semirings, while in Section 3 we introduce basic concepts related to WFAs over semirings. Then, in Section 4, we define two types of simulation matrices and two types of bisimulation matrices for WFAs over positive semirings. They are defined as solutions to particular systems of matrix inequations. We prove that such defined simulations ensure the containment, while bisimulations ensure the equivalence of WFAs. In the consequent section we introduce four types of bisimulation matrices for WFAs over an arbitrary semiring, and prove that they all ensure the equivalence of WFAs. All of them are defined as solutions to particular systems of matrix equations. In the last section, we present the most important general properties of simulations and bisimulations for WFAs over semirings.

2 Semirings and matrices over semirings

In this section, we introduce basic concepts related to semirings, ordered and positive semirings, as well as to matrices over semirings.

Throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers, and for $i, j \in \mathbb{N}$ such that $i \leqslant j$ we use the notation $[i..j] = \{k \in \mathbb{N} \mid i \leqslant k \leqslant j\}$.

An algebraic structure $\mathbb{S} = (S, +, \cdot, 0, 1)$ with a carrier set S, two binary operations $+$ and $\cdot$ on S, and two constants $0, 1 \in S$ is called a *semiring* if the following is true:

(S1) $(S, +, 0)$ is a commutative monoid,
(S2) $(S, \cdot, 1)$ is a monoid,
(S3) the distributivity laws $(r + s) \cdot t = r \cdot t + s \cdot t$ and $t \cdot (r + s) = t \cdot r + t \cdot s$ hold for all $r, s, t \in S$,
(S4) $0 \cdot s = s \cdot 0 = 0$ for every $s \in S$.

We call 0 the *zero* and 1 the *identity* of the semiring $\mathbb{S}$. Note that if $1 = 0$, then $s = 0$ for every $s \in S$, whence $S = \{0\}$. In order to avoid this trivial case, we will assume that all semirings under consideration are non-trivial, i.e., that $1 \neq 0$.

Given a semiring $\mathbb{S} = (S, +, \cdot, 0, 1)$ and natural numbers $m, n \in \mathbb{N}$. A *matrix* of type $m \times n$, or an $m \times n$*-matrix*, with entries in $\mathbb{S}$ (or over $\mathbb{S}$) is defined as a mapping $M : [1..m] \times [1..n] \to S$. For each pair $(i, j) \in [1..m] \times [1..n]$ the value $M(i, j)$ is called the (i, j)-entry of a matrix M. The set of all matrices of type $m \times n$ with entries in a semiring $\mathbb{S}$ is denoted by $\mathbb{S}^{m\times n}$. Similarly, A *vector* of length n with entries in $\mathbb{S}$ (or over $\mathbb{S}$) is defined as a mapping $\nu : [1..n] \to S$. For each $i \in [1..n]$ the value $\nu(i)$ is called the ith entry or ith coordinate of a vector ν. The set of all vectors of length n with entries in a semiring $\mathbb{S}$ is denoted by $\mathbb{S}^n$.

The *zero matrix* of type $m \times n$, denoted by $O_{m\times n}$, is a matrix of type $m \times n$ whose all entries are 0. Similarly, the *zero vector* of length n, denoted by o_n, is a vector of length n whose all entries are 0. For each $n \in \mathbb{N}$, a matrix of type $n \times n$ is called a *square matrix* of order n. The *identity matrix* of order n, denoted by I_n, is a square matrix of order n which satisfies $I_n(i, i) = 1$, for each $i \in [1..n]$, and $I_n(i, j) = 0$, for all $i, j \in [1..n]$ such that $i \neq j$. The transpose of a matrix M is denoted by $M^\top$.

As with classical matrices over a field, for all pairs of matrices from $\mathbb{S}^{m\times n}$ the *matrix addition* is defined pointwise: $(M + N)(i, j) = M(i, j) + N(i, j)$, for all $M, N \in \mathbb{S}^{m\times n}$, $i \in [1..m]$ and $j \in [1..n]$. It is an associative and commutative operation on $\mathbb{S}^{m\times n}$, and in particular, $(\mathbb{S}^{m\times n}, +, O_{m\times n})$ form a commutative monoid. The *matrix product* is defined between matrices from $\mathbb{S}^{m\times n}$ and $\mathbb{S}^{n\times p}$ as follows: for $M \in \mathbb{S}^{m\times n}$ and $N \in \mathbb{S}^{n\times p}$ their product is a matrix $M \cdot N \in \mathbb{S}^{m\times p}$ with entries given by

$$M \cdot N(i, k) = \sum_{j=1}^{n} M(i, j) \cdot M(j, k), \tag{1}$$

for all $(i, j) \in [1..m] \times [1..p]$. The matrix product is associative whenever it is defined, i.e.,

$$(M \cdot N) \cdot P = M \cdot (N \cdot P),$$

for all $M \in \mathbb{S}^{m\times n}$, $N \in \mathbb{S}^{n\times p}$ and $P \in \mathbb{S}^{p\times q}$. In particular, $(\mathbb{S}^{n\times n}, +, \cdot, O_{n\times n}, I_n)$ is a semiring.

Given a matrix $M \in \mathbb{S}^{m\times n}$ and vectors $\mu \in \mathbb{S}^m$ and $\nu \in \mathbb{S}^n$. When μ is treated as a matrix of type $1 \times m$ (row vector) and ν as a matrix of type $n \times 1$ (column vector), the *vector-matrix product* $\mu \cdot M$ and the *matrix-vector product* $M \cdot \nu$ are defined as matrix products. The *scalar product* or *dot product* of vectors $\mu, \nu \in \mathbb{S}^n$ is an element of $\mathbb{S}$ given by

$$\mu \cdot \nu = \sum_{i=1}^{n} \mu(i) \cdot \nu(i). \tag{2}$$

An *ordered semiring* is an algebraic system $\mathbb{S} = (S, +, \cdot, 0, 1, \leqslant)$, where $(S, +, \cdot, 0, 1)$ is a semiring provided with an additional partial order $\leqslant$ on S that satisfies the following conditions:

(OS1) If $r \leqslant s$ then $r + t \leqslant s + t$, for all $r, s, t \in S$,
(OS2) If $r \leqslant s$ and $t \geqslant 0$ then $r \cdot t \leqslant s \cdot t$ and $t \cdot r \leqslant t \cdot s$, for all $r, s, t \in S$.

In other words, condition (OS1) says that the partial order $\leqslant$ is compatible with addition, whereas

(OS2) says that it is compatible with multiplication by positive elements, where by a *positive element* we mean any element $s \in S$ for which $s \geqslant 0$. Clearly, s is positive if and only if $s + r \geqslant r$ for every $r \in S$. The set of all positive elements of an ordered semiring $\mathbb{S}$ is denoted by $\mathbb{S}_+$ and called the *positive cone* od $\mathbb{S}$. It is non-empty, since $0 \in \mathbb{S}_+$, and it is closed under addition and multiplication. Therefore, $\mathbb{S}_+$ is a subsemiring od $\mathbb{S}$ if and only if $1 \in \mathbb{S}_+$, that is, if and only if $1 > 0$. The non-zero elements of $\mathbb{S}_+$ are called *strictly positive*. If $\mathbb{S}_+ = \mathbb{S}$, then $\mathbb{S}$ is said to be a *positive semiring*.

Example 1 (Semirings of numbers) Typical example of semirings is the *semiring of natural numbers* $\mathbb{N}_0$ (with the zero adjoined). This semiring is also ordered, with respect to the usual ordering of natural numbers, and it is also a typical example of a positive semiring.

The *ring of integers* $\mathbb{Z}$, the *field of rational numbers* $\mathbb{Q}$ and the *field of real numbers* $\mathbb{R}$ are certainly semirings, but they are not positive. However, their positive cones $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ are positive semirings.

Example 2 (Additively idempotent semirings) A semiring $\mathbb{S}$ is called *additively idempotent* if the addition in $\mathbb{S}$ is an idempotent operation, i.e., if $s + s = s$, for every $s \in S$. It is well known that this condition is equivalent to the much simpler condition $1 + 1 = 1$.

On an additively idempotent semiring $\mathbb{S}$ we can define an ordering by $s \leqslant t$ if and only if $s + t = t$, for arbitrary $s, t \in S$ (or, equivalently, if and only if $s + u = t$, for some $u \in S$). Such defined ordering is called the *canonical ordering* on an additively idempotent semiring. With respect to this ordering, $\mathbb{S}$ is an ordered semiring, and besides, it is a positive semiring. Moreover, the additive monoid of $\mathbb{S}$ is a join semilattice with $s \vee t = s + t$, for all $s, t \in S$.

There are many important particular examples of additively idempotent semirings. We list the most important of them.

(1) Semirings

$$\mathbb{N}_{\min} = (\mathbb{N}_0 \cup \{\infty\}, \min, +, \infty, 0) \text{ and}$$
$$\mathbb{R}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$$

are called *tropical semirings*, and semirings

$$\mathbb{N}_{\max} = (\mathbb{N}_0 \cup \{-\infty\}, \max, +, -\infty, 0) \text{ and}$$
$$\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$$

are called *arctic semirings*. Note that $\mathbb{N}_{\min}$ is isomorphic to $\mathbb{N}_{\max}$ and $\mathbb{R}_{\min}$ is isomorphic to $\mathbb{R}_{\max}$. The semiring $\mathbb{R}_{\min}$ is also known as the *min-plus algebra* or the *min-plus semiring*, and $\mathbb{R}_{\max}$ is known as the *max-plus algebra* or the *max-plus semiring*.

(2) The semiring $([0, 1], \max, \cdot, 0, 1)$ is known as the *Viterbi semiring* or the *probabilistic semiring*.

(3) Let $\mathbb{L} = (L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice, where $\vee$ denotes the join (supremum) operation, $\wedge$ the meet (infimum) operation, 0 the least element and 1 the greatest element. Then $\mathbb{L}$ is a semiring with addition $\vee$, multiplication $\wedge$, zero 0 and identity 1. Clearly, this semiring is both additively and multiplicatively idempotent.

It is also possible to consider $\wedge$ as addition, $\vee$ as multiplication, 1 as the zero and 0 as the identity, but then we get the same semiring obtained as above from the opposite lattice $\mathbb{L}^{\mathrm{op}}$ of $\mathbb{L}$.

(4) A concrete example of a semiring obtained from a bounded distributive lattice as in (3) is the *Gödel semiring* $\mathbb{I} = ([0, 1], \vee, \wedge, 0, 1)$.

(5) Every two-element additively idempotent semiring is isomorphic to the semiring $(\{0, 1\}, \vee, \wedge, 0, 1)$, that is called the *Boolean semiring*.

(6) Semiring-reducts of semilattice ordered monoids, unital quantales and complete residuated lattices, as well as *Brouwerian lattices* are additively idempotent semirings.

(7) The semiring $(2^{X^*}, \cup, \cdot, \emptyset, \varepsilon)$ of all formal languages over an alphabet X (where $\cdot$ denotes the usual concatenation of languages) is an additively idempotent semiring with set inclusion as its natural ordering.

(8) The semiring $(2^{A\times A}, \cup, \circ, \emptyset, \Delta)$ of all binary relations on a set A (where $\circ$ denotes the usual composition of relations and Δ is the equality relation) is an additively idempotent semiring with relation (set) inclusion as its natural ordering.

For more information on additively idempotent semirings and their important applications the readers are referred to [1, 3, 8, 11, 24, 25, 27].

Example 3 (Lattice-ordered semirings) A semiring $\mathbb{S}$ is lattice-ordered if it has the structure of a lattice, and for arbitrary $s, t \in S$ the following is true:

(LOS1) $s + t = s \vee t$,
(LOS2) $st \leqslant s \wedge t$,

where $\leqslant$ denotes the lattice ordering – the ordering in the underlying lattice, and $\vee$ and $\wedge$ denote the join and meet operations in that lattice. Any lattice-ordered semiring is an ordered semiring, in the sense of the definition above, with respect to the lattice ordering. Moreover, any lattice-ordered semiring is an additively idempotent semiring whose canonical ordering coincides with the lattice ordering. Consequently, any lattice-ordered semiring is positive, and it can be easily verified that $s \leqslant 1$, for every $s \in S$, i.e., 1 is the greatest element.

A lattice-ordered semiring $\mathbb{S}$ is a *complete-lattice-ordered semiring* if $(\mathbb{S}, \vee, \wedge)$ is a complete lattice. A complete-lattice-ordered semiring $\mathbb{S}$ is called a *quantalic lattice-ordered semiring* if multiplication distributes over arbitrary joins from both sides, i.e.,

$$\Big(\bigvee_{i\in I} s_i\Big)\cdot t = \bigvee_{i\in I} s_i\cdot t, \qquad t\cdot\Big(\bigvee_{i\in I} s_i\Big) = \bigvee_{i\in I} t\cdot s_i, \quad (3)$$

for all $t, s_i \in \mathbb{S}$ $(i \in I)$.

Let $\mathbb{S}$ be an ordered semiring and $m, n \in \mathbb{N}$. The ordering on $\mathbb{S}$ can be extended coordinatewise to an ordering on matrices over $\mathbb{S}$ as follows: for arbitrary $M, N \in S^{m\times n}$ we set

$$M \leqslant N \quad \Leftrightarrow \quad M(i,j) \leqslant N(i,j), \qquad (4)$$

for all $(i,j) \in [1..m] \times [1..n]$. This ordering has the following properties

(MO1) If $M \leqslant N$ in $\mathbb{S}^{m\times n}$ then $M+P \leqslant N+P$, for all $M, N, P \in S^{m\times n}$,

(MO2a) If $M \leqslant N$ in $\mathbb{S}^{m\times n}$ and $P \geqslant O_{n\times p}$ in $\mathbb{S}^{n\times p}$ then $M\cdot P \leqslant N\cdot P$, for all $M, N \in \mathbb{S}^{m\times n}$ and $P \in \mathbb{S}^{n\times p}$,

(MO2b) If $M \leqslant N$ in $\mathbb{S}^{m\times n}$ and $Q \geqslant O_{k\times m}$ in $\mathbb{S}^{k\times m}$ then $Q\cdot M \leqslant Q\cdot N$, for all $M, N \in \mathbb{S}^{m\times n}$ and $Q \in \mathbb{S}^{k\times m}$.

In the case of square matrices of order n this means that $(\mathbb{S}^{n\times n}, +, \cdot, O_{n\times n}, I_n)$ is an ordered semiring. If $\mathbb{S}$ is a positive semiring, then the semiring od $n\times n$-matrices $\mathbb{S}^{n\times n}$ is also positive.

For more information on semirings and matrices over semirings we refer to [7, 20, 23].

3 Weighted finite automata over a semiring

Here we introduce basic notions and notation related to weighted finite automata over semirings.

Let X be a finite non-empty set that we call an *alphabet*. The set of all finite sequences of elements of X, including the empty sequence, equipped with the concatenation operation is called the free monoid over X and is denoted by X^*. These sequences are called *words* over X, while the empty sequence ε is called the *empty word*. The set of all non-empty words over X, equipped with the concatenation operation, is called the free semigroup over X and denoted by X^+.

In the sequel, given an alphabet X and a semiring $\mathbb{S} = (S, +, \cdot, 0, 1)$. A function $\varphi : X^* \to \mathbb{S}$ is called a *word function*, or more precisely, an $\mathbb{S}$*-valued word function*. In the literature, such a function is also called a *formal power series* over X and $\mathbb{S}$, for short a *series*, or a *weighted language* in X^* with weights taken in $\mathbb{S}$.

A *weighted finite automaton* (for short: *WFA*) over X and $\mathbb{S}$ is a quadruple $\mathcal{A} = (A, \sigma, \delta, \tau)$ consisting of a finite non-empty *set of states* A, an *initial weight function* $\sigma : A \to \mathbb{S}$, a *weighted transition function* $\delta : A \times X \times A \to \mathbb{S}$, and a *final weight function* $\tau : A \to \mathbb{S}$. In this case X is called the *input alphabet* of the automaton $\mathcal{A}$ and its elements are called *input letters*.

The *behavior* of the weighted finite automaton $\mathcal{A}$ is a word function $[\![\mathcal{A}]\!] : X^* \to \mathbb{S}$ defined by

$$[\![\mathcal{A}]\!](u) = \sum_{(a_0,a_1,\dots,a_k)\in A^{k+1}} \sigma(a_0)\cdot\delta(a_0,x_1,a_1)\cdot \quad (5)$$
$$\cdot\,\delta(a_1,x_2,a_2)\cdot\ldots\cdot\delta(a_{k-1},x_k,a_k)\cdot\tau(a_k),$$

where $u = x_1x_2\dots x_k \in X^*$, $x_1, x_2, \dots, x_k \in X$. We say that the weighted finite automaton $\mathcal{A}$ *computes* the word function $[\![\mathcal{A}]\!]$.

A sequence $(a_0, a_1, \dots, a_k) \in A^{k+1}$ is often called a *run* of $\mathcal{A}$ on the word $u = x_1x_2\dots x_k$, and the product $\sigma(a_0)\cdot\delta(a_0,x_1,a_1)\cdot\delta(a_1,x_2,a_2)\cdot\ldots\cdot\delta(a_{k-1},x_k,a_k)\cdot\tau(a_k)$ is the weight of that run. Therefore, $[\![\mathcal{A}]\!](u)$ is defined as the sum of weights of all runs of $\mathcal{A}$ on the word u.

It is well-known that the class of word functions that can be computed by a weighted finite automaton is the class of rational functions, i.e., the class of all word functions obtained from polynomials using the operations of addition, multiplication, scalar multiplication and Kleene star operation finitely many times (cf. [7]). Recall that a polynomial is a word function taking only finitely many non-zero values.

Let $\mathcal{A} = (A, \sigma, \delta, \tau)$ be a WFA with n states. Assume that $A = \{a_1, a_2, \dots, a_n)$. Then σ and τ can be treated as vectors in $\mathbb{S}^n$, and when we do so we call σ an *initial weight vector* and τ a *final weight vector*. We will treat σ as a row vector and τ as a column vector. We also define a family of *transition matrices* $\{M_x\}_{x\in X} \subseteq \mathbb{S}^{n\times n}$ so that $M_x(i,j) = \delta(a_i, x, a_j)$, for all $x \in X$, $i,j \in [1..n]$. Then we can also represent $\mathcal{A}$ as a quadruple $\mathcal{A} = (n, \sigma, \{M_x\}_{x\in X}, \tau)$, and this representation is called the *linear representation* or *linear form* of the WFA $\mathcal{A}$. The number of states of a WFA $\mathcal{A}$ is also called the *dimension* of $\mathcal{A}$. Matrices M_x ($x \in X$) are called *basic transition matrices*.

For an arbitrary $u = x_1x_2\dots x_k \in X^+$ with $x_1, x_2, \dots, x_k \in X$, a *compound transition matrix* M_u ($u \in X^+$) is defined as follows:

$$M_u = M_{x_1} \cdot M_{x_2} \cdot \dots \cdot M_{x_k}. \tag{6}$$

For the empty word ε, we set that M_ε is the identity matrix of order n (recall that n is the number of states of the automaton $\mathcal{A}$).

It is clear that the behavior of the weighted finite automaton $\mathcal{A}$ can also be specified with

$$\begin{aligned} [\![\mathcal{A}]\!](u) &= \sigma \cdot M_{x_1} \cdot M_{x_2} \cdot \dots \cdot M_{x_k} \cdot \tau \\ &= \sigma \cdot M_u \cdot \tau, \end{aligned} \tag{7}$$

where $u = x_1x_2\dots x_k$, $x_1, x_2, \dots, x_k \in X$, while

$$[\![\mathcal{A}]\!](\varepsilon) = \sigma \cdot \tau.$$

In cases when we deal with several different automata, in order to differentiate them we write the superscripts. For example, the above automaton $\mathcal{A}$ would be denoted by $\mathcal{A} = (A, \sigma^A, \delta^A, \tau^A)$, its linear representation would be given by $\mathcal{A} = (n, \sigma^A, \{M_x^A\}_{x\in X}, \tau^A)$, and its compound transition matrices would be denoted by M_u^A, for all $u \in X^*$.

4 Simulations and bisimulations defined by matrix inequations

In this section we introduce two types of simulations and two types of bisimulations for WFAs over an ordered semiring by means of particular systems of matrix inequations. When the underlying ordered semiring is positive, we prove that these simulations ensure the containment, while bisimulations ensure the equivalence of WFAs.

Let $\mathbb{S} = (S, +, \cdot, 0, 1, \leqslant)$ be an ordered semiring, and let $\mathcal{A}$ and $\mathcal{B}$ be two weighted finite automata over $\mathbb{S}$ of dimensions m and n, given by their linear representations

$\mathcal{A} = (m, \sigma^A, \{M_x^A\}_{x\in X}, \tau^A)$, and
$\mathcal{B} = (n, \sigma^B, \{M_x^B\}_{x\in X}, \tau^B)$.

A *forward simulation* between $\mathcal{A}$ and $\mathcal{B}$ is defined as any non-zero matrix which is a solution to the following system of matrix inequations with the unknown matrix U taking values in $\mathbb{S}^{m\times n}$:

(fs-1) $\sigma^A \leqslant \sigma^B \cdot U^\top$

(fs-2) $U^\top \cdot M_x^A \leqslant M_x^B \cdot U^\top \quad (x \in X)$

(fs-3) $U^\top \cdot \tau^A \leqslant \tau^B$

while any non-zero matrix which is a solution to the system given below is called a *backward simulation* between $\mathcal{A}$ and $\mathcal{B}$:

(bs-1) $\tau^A \leqslant U \cdot \tau^B$

(bs-2) $M_x^A \cdot U \leqslant U \cdot M_x^B \quad (x \in X)$

(bs-3) $\sigma^A \cdot U \leqslant \sigma^B$

Next, if a matrix $U \in \mathbb{S}^{m\times n}$ is a forward simulation between $\mathcal{A}$ and $\mathcal{B}$ and its transpose $U^\top$ is a forward simulation between $\mathcal{B}$ and $\mathcal{A}$, then U is called a *forward bisimulation* between $\mathcal{A}$ and $\mathcal{B}$. In other words, a forward bisimulation between $\mathcal{A}$ and $\mathcal{B}$ is a non-zero solution to the following system of matrix inequations:

(fb-1) $\sigma^A \leqslant \sigma^B \cdot U^\top, \;\; \sigma^B \leqslant \sigma^A \cdot U$

(fb-2) $U^\top \cdot M_x^A \leqslant M_x^B \cdot U^\top, \;\; U \cdot M_x^B \leqslant M_x^A \cdot U$ $(x \in X)$

(fb-3) $U^\top \cdot \tau^A \leqslant \tau^B, \;\; U \cdot \tau^B \leqslant \tau^A$

Similarly, if $U \in \mathbb{S}^{m\times n}$ is a backward simulation between $\mathcal{A}$ and $\mathcal{B}$ and its transpose is a backward simulation between $\mathcal{B}$ and $\mathcal{A}$, then U is called a *backward bisimulation* between $\mathcal{A}$ and $\mathcal{B}$. In other words, a backward bisimulation between $\mathcal{A}$ and $\mathcal{B}$ is a non-zero solution to the following system:

(bb-1) $\tau^A \leqslant U \cdot \tau^B, \;\; \tau^B \leqslant U^\top \cdot \tau^A$

(bb-2) $M_x^A \cdot U \leqslant U \cdot M_x^B, \;\; M_x^B \cdot U^\top \leqslant U^\top \cdot M_x^A$ $(x \in X)$

(bb-3) $\sigma^A \cdot U \leqslant \sigma^B, \;\; \sigma^B \cdot U^\top \leqslant \sigma^A$

The main theorem of the section is as follows:

Theorem 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be weighted finite automata over a positive semiring $\mathbb{S}$. Then the following assertions are true:*

(a) *If there exists a forward or backward simulation between $\mathcal{A}$ and $\mathcal{B}$, then $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$.*

(b) *If there exists a forward or backward bisimulation between $\mathcal{A}$ and $\mathcal{B}$, then $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.*

Proof We will prove only the part of the assertion (a) that refers to forward simulations. The second part of that assertion, which concerns backward simulations, is proved in a similar way, while the assertion (b) follows directly from (a).

Let U be a forward simulation between $\mathcal{A}$ and $\mathcal{B}$. Let us repeat that the positivity of the semiring $\mathbb{S}$ is used to ensure compatibility of the matrix ordering with all allowed matrix multiplications.

First we prove that

$$U^\top \cdot M_u^A \leqslant M_u^B \cdot U^\top, \tag{8}$$

for every word $u \in X^*$. This will be proved by induction on the length of the word u. Clearly, this is true for all words of the length 0 and 1. Suppose that (8) holds for some $u \in X^*$, and consider an arbitrary letter $x \in X$. According to the induction hypothesis and (fs-2) we get

$$\begin{aligned} U^\top \cdot M_{ux}^A &= U^\top \cdot M_u^A \cdot M_x^A \leqslant M_u^B \cdot U^\top \cdot M_x^A \\ &\leqslant M_u^B \cdot M_x^B \cdot U^\top = M_{ux}^B \cdot U^\top. \end{aligned}$$

This means that (8) holds for ux, and we conclude that it holds for every word from X^*.

Next, according to (fs-1), (8) and (fs-3), we have that

$$\begin{aligned} [\![\mathcal{A}]\!](u) &= \sigma^A \cdot M_u^A \cdot \tau^A \leqslant \sigma^B \cdot U^\top \cdot M_u^A \cdot \tau^A \\ &\leqslant \sigma^B \cdot M_u^B \cdot U^\top \cdot \tau^A \leqslant \sigma^B \cdot M_u^B \cdot \tau^B = [\![\mathcal{B}]\!](u), \end{aligned}$$

for every $u \in X^+$, and in particular,

$$[\![\mathcal{A}]\!](\varepsilon) = \sigma^A \cdot \tau^A \leqslant \sigma^B \cdot U^\top \cdot \tau^A \leqslant \sigma^B \cdot \tau^B = [\![\mathcal{B}]\!](\varepsilon).$$

Thus, $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$, which completes the proof. □

5 Bisimulations defined by matrix equations

In the previous section, we saw that by means of a system of matrix inequations one can define simulations and bisimulations for WFAs over an ordered semiring, whereby they realize their basic function when that semiring is positive.

The question that naturally arises is whether it is possible to define certain types of simulations and bisimulations if the underlying semiring is not positive or ordered. In this section, we show that four types of bisimulations for WFAs over an arbitrary semiring can be defined by means of systems of matrix equations, and that all these bisimulations realize their basic function of ensuring the equivalence of WFAs.

Hence, throughout this section, if not noted otherwise, $\mathbb{S}$ will be an arbitrary semiring.

Non-zero solutions to the system of matrix equations

(fbb-1) $\sigma^A = \sigma^B \cdot U^\top$

(fbb-2) $U^\top \cdot M_x^A = M_x^B \cdot U^\top \quad (x \in X)$

(fbb-3) $U^\top \cdot \tau^A = \tau^B$

are called *forward-backward bisimulations*, while non-zero solutions to the system of matrix equations

(bfb-1) $\tau^A = U \cdot \tau^B$

(bfb-2) $M_x^A \cdot U = U \cdot M_x^B \quad (x \in X)$

(bfb-3) $\sigma^A \cdot U = \sigma^B$

are called *backward-forward bisimulations*.

If $\mathbb{S}$ is an ordered semiring, it is not hard to see that a matrix $U \in \mathbb{S}^{m\times n}$ is a forward-backward bisimulation between $\mathcal{A}$ and $\mathcal{B}$ if and only if U is a forward simulation between $\mathcal{A}$ and $\mathcal{B}$ and its transpose $U^\top$ is a backward simulation between $\mathcal{B}$ and $\mathcal{A}$. Similarly, $U \in \mathbb{S}^{m\times n}$ is a backward-forward bisimulation between $\mathcal{A}$ and $\mathcal{B}$ if and only if U is a backward simulation between $\mathcal{A}$ and $\mathcal{B}$ and its transpose $U^\top$ is a forward simulation between $\mathcal{B}$ and $\mathcal{A}$. This is the reason why the names forward-backward and backward-forward bisimulations are used.

The main result concerning forward-backward and backward-forward bisimulations is given by the following theorem.

Theorem 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be weighted finite automata over an arbitrary semiring. If there is a forward-backward or backward-forward bisimulation between $\mathcal{A}$ and $\mathcal{B}$, then $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.*

Proof We will prove only the case when there exists a backward-forward bisimulation U between $\mathcal{A}$ and $\mathcal{B}$. The case that refers to forward-backward bisimulations one proves symmetricaly. The proof is similar to the proof of Theorem 1, but for the sake of completeness we prove this theorem, too. The key point here is that matrix equality is compatible with all allowed matrix multiplications.

As in the proof of Theorem 1, by induction on the length of a word we first prove that

$$M_u^A \cdot U = U \cdot M_u^B, \tag{9}$$

for every $u \in X^*$. Clearly, this holds for words of lengths 0 and 1. Suppose that (9) holds for some word $u \in X^*$ and consider an arbitrary $x \in X$. According to the induction hypothesis and (bfb-2) we obtain that

$$\begin{aligned} M_{ux}^A \cdot U &= M_u^A \cdot M_x^A \cdot U = M_u^A \cdot U \cdot M_x^B \\ &= U \cdot M_u^B \cdot M_x^B = U \cdot M_{ux}^B. \end{aligned}$$

Therefore, (9) holds for the word ux, and by induction we conclude that it holds for all words from X^*.

Further, according to (9) and (bfb-1)–(bfb-3), for each $u \in X^+$ we have that

$$\begin{aligned} [\![\mathcal{A}]\!](u) &= \sigma^A \cdot M_u^A \cdot \tau^A = \sigma^A \cdot M_u^A \cdot U \cdot \tau^B \\ &= \sigma^A \cdot U \cdot M_u^B \cdot \tau^B = \sigma^B \cdot M_u^B \cdot \tau^B = [\![\mathcal{B}]\!](u), \end{aligned}$$

and also,

$$[\![\mathcal{A}]\!](\varepsilon) = \sigma^A \cdot \tau^A = \sigma^A \cdot U \cdot \tau^B = \sigma^B \cdot \tau^B = [\![\mathcal{B}]\!](\varepsilon).$$

Therefore, $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$, which was to be proved. □

Our next task is to find certain types of bisimulations that can be used as substitutes for forward and backward bisimulations. For this reason, we introduce the following systems of matrix equations with two unknowns:

$$\begin{aligned} &\text{(fb*-1)} \quad \sigma^A \cdot U \cdot V = \sigma^B \cdot V, \quad \sigma^B \cdot V \cdot U = \sigma^A \cdot U \\ &\text{(fb*-2)} \quad M_x^A \cdot U \cdot V = U \cdot M_x^B \cdot V, \\ &\qquad\qquad M_x^B \cdot V \cdot U = V \cdot M_x^A \cdot U \quad (x \in X) \\ &\text{(fb*-3)} \quad \tau^A = U \cdot \tau^B, \quad \tau^B = V \cdot \tau^A \end{aligned}$$

and

$$\begin{aligned} &\text{(bb*-1)} \quad U \cdot V \cdot \tau^A = U \cdot \tau^B, \quad V \cdot U \cdot \tau^B = V \cdot \tau^A \\ &\text{(bb*-2)} \quad U \cdot V \cdot M_x^A = U \cdot M_x^B \cdot V, \\ &\qquad\qquad V \cdot U \cdot M_x^B = V \cdot M_x^A \cdot U \quad (x \in X) \\ &\text{(bb*-3)} \quad \sigma^A = \sigma^B \cdot V, \quad \sigma^B = \sigma^A \cdot U \end{aligned}$$

as well as the system

$$\text{(MP-12)} \quad U \cdot V \cdot U = U, \quad V \cdot U \cdot V = V$$

where U is an unknown matrix taking values in $\mathbb{S}^{m\times n}$ and V is an unknown matrix taking values in $\mathbb{S}^{n\times m}$. The reason why we use notation "fb*" and "bb*" will be explained later.

The equation $U \cdot V \cdot U = U$ is known as the first Moore-Penrose equation, and $V \cdot U \cdot V = V$ is known as the second Moore-Penrose equation, which is why we denote these equations together by (MP-12).

Here we also prove that the solutions of the considered systems, if they exist, ensure the equivalence of automata.

Theorem 3 *Let $\mathcal{A}$ and $\mathcal{B}$ be weighted finite automata of dimensions m and n over a semiring $\mathbb{S}$.*

If there exists a pair $U \in \mathbb{S}^{m\times n}$ and $V \in \mathbb{S}^{n\times m}$ that is a solution to any of the systems (fb*-1)–(fb*-3) *and* (MP-12)*, and* (bb*-1)–(bb*-3) *and* (MP-12)*, then* $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$.

Proof We will prove only the case when U and V form a solution to the system (fb*-1)–(fb*-3) and (MP-12). The case concerning the system (bb*-1)-(bb*-3) and (MP-12) is proved in a symmetrical way.

Let us assume that U and V form a solution to system (fb*-1)–(fb*-3) and (MP-12). First we prove that

$$M_u^A \cdot U \cdot V = U \cdot M_u^B \cdot V, \quad M_u^B \cdot V \cdot U = V \cdot M_u^A \cdot U, \tag{10}$$

for every word $u \in X^*$. This will be proved by induction on the length of u. It is clear that (10) holds for $u = \varepsilon$ and $u = x \in X$. Suppose that (10) holds for some word $u \in X^*$ and consider an arbitrary letter $x \in X$. Using the induction hypothesis, (fb*-2) and (MP-12) we get

$$\begin{aligned} M_{ux}^A \cdot U \cdot V &= M_u^A \cdot M_x^A \cdot U \cdot V = M_u^A \cdot U \cdot M_x^B \cdot V \\ &= M_u^A \cdot U \cdot V \cdot U \cdot M_x^B \cdot V \\ &= U \cdot M_u^B \cdot V \cdot U \cdot M_x^B \cdot V \\ &= U \cdot M_u^B \cdot V \cdot M_x^A \cdot U \cdot V \\ &= U \cdot M_u^B \cdot M_x^B \cdot V \cdot U \cdot V = U \cdot M_{ux}^B \cdot V. \end{aligned}$$

In a similar way we prove that $M_{ux}^B \cdot V \cdot U = V \cdot M_{ux}^A \cdot U$. Therefore, (10) holds for the word ux, and by induction we conclude that it holds for every word from X^*.

Next, according to (fb*-1), (10), (fb*-3) and (MP-12) we have that

$$\begin{aligned} [\![\mathcal{A}]\!](u) &= \sigma^A \cdot M_u^A \cdot \tau^A = \sigma^A \cdot M_u^A \cdot U \cdot \tau^B \\ &= \sigma^A \cdot M_u^A \cdot U \cdot V \cdot U \cdot \tau^B \\ &= \sigma^A \cdot U \cdot M_u^B \cdot V \cdot U \cdot \tau^B \\ &= \sigma^A \cdot U \cdot V \cdot U \cdot M_u^B \cdot V \cdot U \cdot \tau^B \\ &= \sigma^B \cdot V \cdot U \cdot M_u^B \cdot V \cdot U \cdot \tau^B \\ &= \sigma^B \cdot V \cdot M_u^A \cdot U \cdot V \cdot U \cdot \tau^B \end{aligned}$$

$$= \sigma^B \cdot V \cdot M_u^A \cdot U \cdot \tau^B = \sigma^B \cdot M_u^B \cdot V \cdot U \cdot \tau^B$$
$$= \sigma^B \cdot M_u^B \cdot V \cdot \tau^A = \sigma^B \cdot M_u^B \cdot \tau^B$$
$$= [\![\mathcal{B}]\!](u),$$

and

$$[\![\mathcal{A}]\!](\varepsilon) = \sigma^A \cdot \tau^A = \sigma^A \cdot U \cdot \tau^B$$
$$= \sigma^A \cdot U \cdot V \cdot U \cdot \tau^B = \sigma^B \cdot V \cdot U \cdot \tau^B$$
$$= \sigma^B \cdot V \cdot \tau^A = \sigma^B \cdot \tau^B = [\![\mathcal{B}]\!](\varepsilon).$$

Therefore, $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$, which completes the proof of the theorem. □

Example 4 Let $\mathcal{A}$ and $\mathcal{B}$ be weighted finite automata of dimensions 3 and 2 over the Gödel semiring $\mathbb{I}$ and the two-element alphabet $X = \{x, y\}$, given by the following initial and final weight vectors and transition matrices:

$$\sigma^A = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad M_x^A = \begin{bmatrix} 1 & 0.3 & 0.4 \\ 0.5 & 1 & 0.3 \\ 0.4 & 0.6 & 0.7 \end{bmatrix},$$
$$M_y^A = \begin{bmatrix} 0.5 & 0.6 & 0.2 \\ 0.6 & 0.3 & 0.4 \\ 0.7 & 0.7 & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$
$$\sigma^B = \begin{bmatrix} 1 & 0.5 \end{bmatrix}, \quad M_x^B = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.7 \end{bmatrix},$$
$$M_y^B = \begin{bmatrix} 0.6 & 0.6 \\ 0.7 & 1 \end{bmatrix}, \quad \tau^B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

By a straightforward check we find that the pair of matrices

$$U = \begin{bmatrix} 1 & 0.6 \\ 1 & 0.6 \\ 0.6 & 1 \end{bmatrix}, \quad V = U^\top,$$

is a solution to the system consisting of (fb*-1)–(fb*-3) and (MP-12).

On the other hand, systems (fbb-1)–(fbb-3) and (bfb-1)–(bfb-3) have no solutions. Indeed, suppose that a matrix $U \in \mathbb{I}^{3\times 2}$ satisfies (fbb-1) and (fbb-3). From $\sigma^B \cdot U^\top = \sigma^A$ we conclude that U has the form

$$U = \begin{bmatrix} 0 & 0 \\ 1 & s \\ 0 & 0 \end{bmatrix}$$

for some $s \in \mathbb{I}$, and from $U^\top \cdot \tau^A = \tau^B$ we conclude that $s = 1$. However, then

$$U^\top \cdot M_x^A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0.3 & 0.4 \\ 0.5 & 1 & 0.3 \\ 0.4 & 0.6 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0.3 \\ 0.5 & 1 & 0.3 \end{bmatrix}$$
$$\neq \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.7 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = M_x^B \cdot U^\top,$$

which means that the system (fbb-1)–(fbb-3) has no solutions.

Further, suppose that $U \in \mathbb{I}^{3\times 2}$ is a matrix which satisfies (bfb-3). From this we get that $r_2^U = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$, whence it follows that

$$(M_x^A \cdot U)(2,2) = r_2^{M_x^A} \cdot c_2^U = \begin{bmatrix} 0.5 & 1 & 0.3 \end{bmatrix} \cdot \begin{bmatrix} u_{12} \\ 0.5 \\ u_{32} \end{bmatrix}$$
$$= (0.5 \wedge u_{12}) \vee 0.5 \vee (0.3 \wedge u_{32}) = 0.5,$$

(where $u_{ij} = U(i,j)$), while

$$(U \cdot M_x^B)(2,2) = r_2^U \cdot c_2^{M_x^B} = \begin{bmatrix} 1 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0.6 \\ 0.7 \end{bmatrix} = 0.6.$$

Therefore, $M_x^A \cdot U \neq U \cdot M_x^B$, and we conclude that the system (bfb-1)–(bfb-3) has no solutions.

In a similar way we can show that the system consisting of (bb*-1)–(bb*-3) and (MP-12) does not have solutions. Indeed, if $U \in \mathbb{I}^{3\times 2}$ and $V \in \mathbb{I}^{2\times 3}$ is a pair of matrices which satisfies (bb*-3), then $r_2^U = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$ and $r_1^V = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, so we get

$$(V \cdot M_x^A \cdot U)(1,2) = 0.5 \neq 0.6 = (V \cdot U \cdot M_x^B)(1,2),$$

and thus, the system consisting of (bb*-1)–(bb*-3) and (MP-12) has no solution.

The next theorem establishes a connection between forward and backward bisimulations for WFAs over positive semirings and solutions to the systems (fb*-1)–(fb*-3) and (bb*-1)–(bb*-3)).

Theorem 4 *Let $\mathcal{A}$ and $\mathcal{B}$ be weighted finite automata of dimensions m and n over a positive semiring $\mathbb{S}$, and let $U \in \mathbb{S}^{m\times n}$ be a matrix satisfying*

$$U \cdot U^\top \cdot U \leqslant U, \quad I_m \leqslant U \cdot U^\top, \quad I_n \leqslant U^\top \cdot U. \quad (11)$$

Then U is a forward bisimulation (*resp. backward bisimulation*) *between $\mathcal{A}$ and $\mathcal{B}$ if and only if U and $U^\top$ form a solution to system* (fb*-1)–(fb*-3) (*resp. system* (bb*-1)–(bb*-3)).

Proof We will prove only the assertion that refers to forward bisimulations and the system (fb*-1)–(fb*-3). The assertion that refers to backward bisimulations and the system (bb*-1)–(bb*-3) is proved similarly.

Let U be a forward bisimulation. By (fb-1) and (11) it follows that

$$\sigma^A \cdot U \cdot U^\top \leqslant \sigma^B \cdot U^\top \cdot U \cdot U^\top \leqslant \sigma^B \cdot U^\top \leqslant \sigma^A \cdot U \cdot U^\top,$$

whence $\sigma^A \cdot U \cdot U^\top = \sigma^B \cdot U^\top$. In the same way we show that $\sigma^B \cdot U^\top \cdot U = \sigma^A \cdot U$. Therefore, U and $U^\top$ satisfy (fb*-1).

Further, according to (fb-2) and (11) we get

$$M_x^A \cdot U \cdot U^\top = I_m \cdot M_x^A \cdot U \cdot U^\top$$
$$\leqslant U \cdot U^\top \cdot M_x^A \cdot U \cdot U^\top \leqslant U \cdot M_x^B \cdot U^\top \cdot U \cdot U^\top$$
$$\leqslant U \cdot M_x^B \cdot U^\top \leqslant M_x^A \cdot U \cdot U^\top,$$

and thus, U and $U^\top$ satisfy the first equality in (fb*-2). In a similar way we prove that they satisfy the second equality in (fb*-2).

Finally, from (fb-3) and (11) we obtain

$$U \cdot \tau^B \leqslant \tau^A = I_m \cdot \tau^A \leqslant U \cdot U^\top \cdot \tau^A \leqslant U \cdot \tau^B,$$

and therefore, U satisfies the first equality in (fb*-3). Similarly we prove that $U^\top$ satisfies the second equality in (fb*-3).

Conversely, let U and $U^\top$ form a solution to system (fb*-1)–(fb*-3). Then

$$\sigma^A = \sigma^A \cdot I_m \leqslant \sigma^A \cdot U \cdot U^\top = \sigma^B \cdot U^\top,$$

and similarly, $\sigma^B \leqslant \sigma^A \cdot U$. Further,

$$\begin{aligned} U^\top \cdot M_x^A &= U^\top \cdot M_x^A \cdot I_m \leqslant U^\top \cdot M_x^A \cdot U \cdot U^\top \\ &= M_x^B \cdot U^\top \cdot U \cdot U^\top \leqslant M_x^B \cdot U^\top, \end{aligned}$$

and similarly, $U \cdot M_x^B \leqslant M_x^A \cdot U$. Finally, (fb-3) follows directly from (fb*-3). □

Remark 1 Let us note that, due to properties of the transpose, condition $U \cdot U^\top \cdot U \leqslant U$ implies symmetric condition $U^\top \cdot U \cdot U^\top \leqslant U^\top$, and vice versa. For this reason, this second condition is omitted from (11).

On the other hand, if the second or third condition from (11) holds, i.e., if if $I_m \leqslant U \cdot U^\top$ or $I_n \leqslant U^\top \cdot U$, then $U \leqslant U \cdot U^\top \cdot U$, and thus, condition $U \cdot U^\top \cdot U \leqslant U$ in (11) can be replaced by $U \cdot U^\top \cdot U = U$. That also implies $U^\top \cdot U \cdot U^\top = U^\top$.

6 General properties of simulations and bisimulations

In the last section, we present some basic general properties of simulations and bisimulations.

For $\theta \in \{\mathrm{fs}, \mathrm{bs}\}$, a matrix that satisfies (θ-1), (θ-2) and (θ-3) is called a *simulation of type* θ, shortly a θ*-simulation*, and the set of all simulations of type θ between WFAs $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathscr{S}^\theta(\mathcal{A}, \mathcal{B})$.

On the other hand, for $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}\}$, a matrix which satisfies (θ-1), (θ-2) and (θ-3) is called a *bisimulation of type* θ or a θ*-bisimulation*, and the set of all bisimulations of type θ between WFAs $\mathcal{A}$ and $\mathcal{B}$ is denoted by $\mathscr{B}^\theta(\mathcal{A}, \mathcal{B})$. In all the mentioned cases θ is called a *type* of simulations or bisimulations.

If not noted otherwise, it goes without saying that simulations of type $\theta \in \{\mathrm{fs}, \mathrm{bs}\}$ and bisimulations of type $\theta \in \{\mathrm{fb}, \mathrm{bb}\}$ are defined for WFAs over a positive semiring, while bisimulations of type $\theta \in \{\mathrm{fbb}, \mathrm{bfb}\}$ are defined for WFAs over an arbitrary semiring.

Simulations and bisimulations between different weighted finite automata are called *heterogeneous*, and simulations and bisimulations between a weighted finite automaton and itself are called *homogeneous*. Homogeneous simulations are also called *autosimulations*, and homogeneous bisimulations are *autobisimulations*. If $\mathcal{A}$ is a WFA over a positive semiring $\mathbb{S}$ and $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}\}$, for the sake of simplicity we write $\mathscr{B}^\theta(\mathcal{A})$ instead of $\mathscr{B}^\theta(\mathcal{A}, \mathcal{A})$.

Theorem 5 *Let $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}\}$, and let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be WFAs over a semiring $\mathbb{S}$. If $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}\}$, we assume that $\mathbb{S}$ is positive, and if $\theta \in \{\mathrm{fbb}, \mathrm{bfb}\}$, we assume that $\mathbb{S}$ is an arbitrary semiring. Then the following assertions are true:*

(a) *For $\theta \in \{\mathrm{fs}, \mathrm{bs}\}$, if $U \in \mathscr{S}^\theta(\mathcal{A}, \mathcal{C})$ and $V \in \mathscr{S}^\theta(\mathcal{B}, \mathcal{C})$, then $U \cdot V \in \mathscr{S}^\theta(\mathcal{A}, \mathcal{C})$;*

(b) *For any $\theta \in \{\mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}\}$, if $U \in \mathscr{B}^\theta(\mathcal{A}, \mathcal{C})$ and $V \in \mathscr{B}^\theta(\mathcal{B}, \mathcal{C})$, then $U \cdot V \in \mathscr{B}^\theta(\mathcal{A}, \mathcal{C})$.*

Proof We will prove this theorem only in the case (a), for $\theta = \mathrm{fs}$. The case (b) can be proved similarly.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ have dimensions m, n and p, respectively. Then $U \in \mathbb{S}^{m\times n}$, $V \in \mathbb{S}^{n\times p}$ and $U \cdot V \in \mathbb{S}^{m\times p}$. Due to the compatibility of the matrix ordering with all possible matrix products and the facts that U and V satisfy (fs-1)–(fs-3), we obtain that

$$\begin{aligned} &\sigma^A \leqslant \sigma^B \cdot U^\top \leqslant \sigma^C \cdot V^\top \cdot U^\top = \sigma^C \cdot (U \cdot V)^\top, \\ &(U \cdot V)^\top \cdot M_x^A = V^\top \cdot U^\top \cdot M_x^A \leqslant V^\top \cdot M_x^B \cdot U^\top \\ &\quad \leqslant M_x^C \cdot V^\top \cdot U^\top = M_x^C \cdot (U \cdot V)^\top, \text{ for any } x \in X, \\ &(U \cdot V)^\top \cdot \tau^A = V^\top \cdot U^\top \cdot \tau^A \leqslant V^\top \cdot \tau^B \leqslant \tau^C. \end{aligned}$$

This proves that $U \cdot V$ satisfies (fs-1)–(fs-3). □

Theorem 6 *Let $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}, \mathrm{fbb}, \mathrm{bfb}\}$, and let $\mathcal{A}$ and $\mathcal{B}$ be WFAs of dimensions m and n over a semiring $\mathbb{S}$. If $\theta \in \{\mathrm{fbb}, \mathrm{bfb}\}$, we assume that $\mathbb{S}$ is an arbitrary semiring, and if $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}\}$, we assume that $\mathbb{S}$ is a positive semiring. Then the following assertions for matrices $U, V \in \mathbb{S}^{m\times n}$ are true:*

(a) *If U and V satisfy (θ-2), then $U + V$ satisfies (θ-2).*

(b) *If $\theta \in \{\mathrm{fs}, \mathrm{bs}, \mathrm{fb}, \mathrm{bb}\}$ and if U and V satisfy (θ-1), then $U + V$ satisfies (θ-1).*

(c) *If $\theta \in \{\mathrm{fbb}, \mathrm{bfb}\}$ and $\mathbb{S}$ is additively idempotent, and if U and V satisfy (θ-1), then $U + V$ satisfies (θ-1).*

(d) *If $\mathbb{S}$ is additively idempotent and if U and V satisfy (θ-3), then $U + V$ satisfies (θ-3).*

Proof In (a), (b) and (d) we only prove the case related to forward simulations, and in (c) the case related to forward-backward bisimulations. The remaining cases are proved similarly.

(a) Due to the distributivity of matrix multiplication (when it is defined) over matrix sum, the compatibility of matrix ordering with matrix addition, and the assumption that U and V satisfy (fs-2), we have that

$$\begin{aligned}(U+V)^\top\cdot M_x^A &= (U^\top+V^\top)\cdot M_x^A\\ &= U^\top\cdot M_x^A+V^\top\cdot M_x^A \leqslant M_x^B\cdot U^\top+M_x^B\cdot V^\top\\ &= M_x^B\cdot(U^\top+V^\top)=M_x^B\cdot(U+V)^\top,\end{aligned}$$

for every $x\in X$. This proves (b).

(b) According to the compatibility of matrix ordering with matrix sums, the distributivity of matrix multiplication (when it is defined) over matrix sum, and the assumption that U and V satisfy (fs-1) we get

$$\begin{aligned}\sigma^A+\sigma^A &\leqslant \sigma^B\cdot U^\top+\sigma^B\cdot V^\top\\ &= \sigma^B\cdot(U^\top+V^\top)=\sigma^B\cdot(U+V)^\top.\end{aligned}$$

On the other hand, according to the positivity of the underlying semiring $\mathbb{S}$ we get $\sigma^A\leqslant\sigma^A+\sigma^A$, which implies $\sigma^A\leqslant\sigma^B\cdot(U+V)^\top$. Thus, $U+V$ satisfies (fs-1).

(c) The proof of this assertion is almost identical to the proof of (b), the only difference being that all inequalities in (b) have been replaced by equalities. The equality $\sigma^A=\sigma^A+\sigma^A$ follows from the additive idempotency.

(d) Again according to the compatibility of matrix ordering with matrix sums, the distributivity of matrix multiplication (when it is defined) over matrix sum, and the assumption that U and V satisfy (fs-3) we get

$$\begin{aligned}(U+V)^\top\cdot\tau^A &= (U^\top+V^\top)\cdot\tau^A\\ &= U^\top\cdot\tau^A+U^\top\cdot\tau^A\leqslant\tau^A+\tau^A.\end{aligned}$$

On the other hand, according to the additive idempotency of the underlying semiring we get $\tau^A+\tau^A=\tau^A$. Therefore, $U+V$ satisfies (fs-3). □

Theorem 7 *Let $\mathcal{A}$ and $\mathcal{B}$ be WFAs over a quantalic lattice-ordered semiring and $\theta\in\{\mathrm{fs},\mathrm{bs},\mathrm{fb},\mathrm{bb},\mathrm{fbb},\mathrm{bfb}\}$.*

If there is at least one simulation/bisimulation of type θ between $\mathcal{A}$ and $\mathcal{B}$, then there is the greatest simulation/bisimulation of this type between $\mathcal{A}$ and $\mathcal{B}$.

Proof Suppose that there is at least one simulation/bisimulation of type θ between $\mathcal{A}$ and $\mathcal{B}$. Let $\{U_i\}_{i\in I}$ be the family of all simulations/bisimulations of this type between $\mathcal{A}$ and $\mathcal{B}$, and let

$$U=\bigvee_{i\in I}U_i.$$

Using (3), it is not difficult to verify that

$$M_x^A\cdot U=M_x^A\cdot\Big(\bigvee_{i\in I}U_i\Big)=\bigvee_{i\in I}(M_x^A\cdot U_i),$$

$$U\cdot M_x^B=\Big(\bigvee_{i\in I}U_i\Big)\cdot M_x^B=\bigvee_{i\in I}(U_i\cdot M_x^B),$$

$$U^\top\cdot M_x^A=\Big(\bigvee_{i\in I}U_i^\top\Big)\cdot M_x^A=\bigvee_{i\in I}(U_i^\top\cdot M_x^A),$$

$$M_x^B\cdot U^\top=M_x^B\cdot\Big(\bigvee_{i\in I}U_i^\top\Big)=\bigvee_{i\in I}(M_x^B\cdot U_i^\top),$$

for each $x\in X$, and since all U_i's satisfy (θ-2) we conclude that U also satisfies (θ-2). In the same way we prove that U satisfies (θ-1) and (θ-3), and thus, U is a simulation/bisimulation of type θ. It is clear that U is the greatest simulation/bisimulation of this type. □

Theorem 8 *Let $\theta\in\{\mathrm{fb},\mathrm{bb}\}$ and let $\mathcal{A}$ and $\mathcal{B}$ be WFAs over a positive semiring.*

(a) *If there exists the greatest bisimulation U of type θ between $\mathcal{A}$ and $\mathcal{B}$, then it satisfies*

$$U\cdot U^\top\cdot U\leqslant U.$$

(b) *If there exists the greatest autobisimulation U of type θ on $\mathcal{A}$, then it satisfies*

$$U\cdot U\leqslant U\ \text{ and }\ I_m\leqslant U,$$

where m is the dimension of $\mathcal{A}$.

Proof (a) Let U be the greatest bisimulation of type θ between $\mathcal{A}$ and $\mathcal{B}$. Then $U^\top$ is a bisimulation of type θ between $\mathcal{B}$ and $\mathcal{A}$, and according to Theorem 5 we have that $U\cdot U^\top\cdot U$ is a bisimulation of type θ between $\mathcal{A}$ and $\mathcal{B}$. Consequently, $U\cdot U^\top\cdot U\leqslant U$.

(b) Let U be the greatest autobisimulation of type θ on $\mathcal{A}$. As in the proof of (a) we get that $U\cdot U\leqslant U$. Since I_m is also an autobisimulation of type θ on $\mathcal{A}$, we conclude that $I_m\leqslant U$. □

Theorem 9 *Let $\mathcal{A}$ and $\mathcal{B}$ be WFAs of dimensuions m and n over a positive semiring $\mathbb{S}$, and let $U\in\mathbb{S}^{m\times n}$. Then the following assertions are true:*

(a) *If $\theta\in\{\mathrm{fb},\mathrm{bb}\}$ and $U\in\mathscr{B}^\theta(\mathcal{A},\mathcal{B})$, then*

$$U\cdot U^\top\in\mathscr{B}^\theta(\mathcal{A})\ \text{ and }\ U^\top\cdot U\in\mathscr{B}^\theta(\mathcal{B}).$$

(b) *If $U\in\mathscr{B}^{\mathrm{bfb}}(\mathcal{A},\mathcal{B})$ and satisfies (11), then*

$$U\cdot U^\top\in\mathscr{B}^{\mathrm{fb}}(\mathcal{A})\ \text{ and }\ U^\top\cdot U\in\mathscr{B}^{\mathrm{bb}}(\mathcal{B}).$$

(c) *If $U\in\mathscr{B}^{\mathrm{fbb}}(\mathcal{A},\mathcal{B})$ and satisfies (11), then*

$$U\cdot U^\top\in\mathscr{B}^{\mathrm{bb}}(\mathcal{A})\ \text{ and }\ U^\top\cdot U\in\mathscr{B}^{\mathrm{fb}}(\mathcal{B}).$$

Proof (a) We will prove only the case $\theta=\mathrm{fb}$. The case $\theta=\mathrm{bb}$ is proved in a similar way.

Let $U \in \mathscr{B}^{\mathrm{fb}}(\mathcal{A}, \mathcal{B})$. First we have that $\sigma^A \leqslant \sigma^B \cdot U^\top \leqslant \sigma^A \cdot U \cdot U^\top$, which means that $U \cdot U^\top$ satisfies (fb-1) with $A = B$. Next, we get

$$U \cdot U^\top \cdot M_x^A \leqslant U \cdot M_x^B \cdot U^\top \leqslant M_x^A \cdot U \cdot U^\top,$$

and thus, $U \cdot U^\top$ satisfies (fb-2) with $A = B$. Finally, $U \cdot U^\top \cdot \tau^A \leqslant U \cdot \tau^B \leqslant \tau^A$, so $U \cdot U^\top$ also satisfies (fb-3) with $A = B$. Therefore, $U \cdot U^\top \in \mathscr{B}^{\mathrm{fb}}(\mathcal{A})$. In the same manner we prove that $U^\top \cdot U \in \mathscr{B}^{\mathrm{fb}}(\mathcal{B})$.

(b) Let $U \in \mathscr{B}^{\mathrm{bfb}}(\mathcal{A}, \mathcal{B})$. According to (11) we get $\sigma^A = \sigma^A \cdot I_m \leqslant \sigma^A \cdot U \cdot U^\top$, so $U \cdot U^\top$ satisfies (fb-1) with $A = B$. Next, according to (bfb-2) and Remark 1 we get

$$\begin{aligned} M_x^A \cdot U \cdot U^\top &= U \cdot M_x^B \cdot U^\top = U \cdot U^\top \cdot U \cdot M_x^B \cdot U^\top \\ &= U \cdot U^\top \cdot M_x^A \cdot U \cdot U^\top, \end{aligned}$$

whence

$$\begin{aligned} U \cdot U^\top \cdot M_x^A &= U \cdot U^\top \cdot M_x^A \cdot I_m \\ &\leqslant U \cdot U^\top \cdot M_x^A \cdot U \cdot U^\top = M_x^A \cdot U \cdot U^\top. \end{aligned}$$

Therefore, $U \cdot U^\top$ satisfies (fb-2) with $A = B$. Finally,

$$U \cdot U^\top \cdot \tau^A = U \cdot U^\top \cdot U \cdot \tau^B = U \cdot \tau^B = \tau^A,$$

which means that $U \cdot U^\top$ satisfies (fb-3) with $A = B$. We have thus proved that $U \cdot U^\top \in \mathscr{B}^{\mathrm{fb}}(\mathcal{A})$.

In the same way we prove $U^\top \cdot U \in \mathscr{B}^{\mathrm{bb}}(\mathcal{B})$.

(c) This assertion is proved similarly to (b). □

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