Representations of quaternion W-MPCEP, W-CEPMP and W-MPCEPMP inverses

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Abstract

The aim of this research is to introduce and investigate the right and left **W**-MPCEP, **W**-CEPMP and **W**-MPCEPMP generalized inverses for quaternion matrices. These generalized inverses are introduced as extensions of corresponding generalized inverses applicable to complex matrices. Some new characterizations and expressions of these inverses are presented. Determinantal representations of these new inverses are established in terms of noncommutative row-column minors of corresponding matrices. To illustrate our results, a numerical example is given.

Key words and phrases: generalized inverse; Moore-Penrose inverse; core-EP inverse; weighted core-EP inverse; determinantal representation.

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1 Introduction

Let $\mathbb{H} = \{\eta_0 + \eta_1\mathbf{i} + \eta_2\mathbf{j} + \eta_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1,\ \eta_0, \eta_1, \eta_2, \eta_3 \in \mathbb{R}\}$ be the quaternion skew field. For $\eta = \eta_0 + \eta_1\mathbf{i} + \eta_2\mathbf{j} + \eta_3\mathbf{k} \in \mathbb{H}$, the quaternion $\overline{\eta} = \eta_0 - \eta_1\mathbf{i} - \eta_2\mathbf{j} - \eta_3\mathbf{k}$ and the real number $\|\eta\| = \sqrt{\eta\overline{\eta}} = \sqrt{\overline{\eta}\eta} = \sqrt{\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2}$ are the conjugate and norm of η, respectively.

The symbols rank($\mathbf{A}$) and $\mathbf{A}^*$, respectively, stand for the rank and conjugate transpose of $\mathbf{A} \in \mathbb{H}^{m\times n}$, where $\mathbb{H}^{m\times n}$ contains all $m \times n$ matrices on $\mathbb{H}$. Denote by

$$\mathcal{C}_r(\mathbf{A}) = \{\mathbf{s} \in \mathbb{H}^{m\times 1} : \mathbf{s} = \mathbf{At},\ \mathbf{t} \in \mathbb{H}^{n\times 1}\}, \qquad \mathcal{N}_r(\mathbf{A}) = \{\mathbf{t} \in \mathbb{H}^{m\times 1} :\ \mathbf{At} = \mathbf{0}\},$$

$$\mathcal{R}_l(\mathbf{A}) = \{\mathbf{s} \in \mathbb{H}^{1\times n} :\ \mathbf{s} = \mathbf{tA},\ \mathbf{t} \in \mathbb{H}^{1\times m}\} \qquad \text{and} \qquad \mathcal{N}_l(\mathbf{A}) = \{\mathbf{t} \in \mathbb{H}^{1\times n} :\ \mathbf{tA} = \mathbf{0}\}.$$

the right column space, right null space, left row space and left null space of $\mathbf{A}$, respectively. The set $\mathbb{H}_r^{m\times n}$ presents the subset of matrices from $\mathbb{H}^{m\times n}$ of rank r.

For $\mathbf{A} \in \mathbb{H}^{m\times n}$, the unique solution $\mathbf{X} \in \mathbb{H}^{n\times m}$ to the system of equations

$$\mathbf{X} = \mathbf{XAX}, \quad \mathbf{A} = \mathbf{AXA}, \quad \mathbf{AX} = (\mathbf{AX})^*, \quad \mathbf{XA} = (\mathbf{XA})^*,$$

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is *the Moore-Penrose* (or shortly MP) inverse $\mathbf{A}^{\dagger}$ of $\mathbf{A}$ [21,57]. The index of $\mathbf{A} \in \mathbb{H}^{n\times n}$ (denoted by $k = \mathrm{Ind}\,(\mathbf{A})$) is the smallest nonnegative integer such that $\mathrm{rank}(\mathbf{A^{k+1}}) = \mathrm{rank}(\mathbf{A^{k}})$. The *Drazin inverse* $\mathbf{A}^{\mathrm{D}}$ of $\mathbf{A} \in \mathbb{H}^{n\times n}$ with the index $k = \mathrm{Ind}\,(\mathbf{A})$ is the unique matrix for which [22,57]

$$\mathbf{XA} = \mathbf{AX}, \quad \mathbf{X} = \mathbf{XAX}, \quad \mathbf{A}^{k} = \mathbf{XA}^{k+1}.$$

Especially, for $\mathrm{Ind}\,(\mathbf{A}) \leq 1$, $\mathbf{A}^{\mathrm{D}} = \mathbf{A}^{\#}$ reduces to the *group inverse* of $\mathbf{A}$. Some recent applications of generalized inverses can be found in [5,13–15,60].

The core-EP inverse $\mathbf{A}^{\textcircled{\dagger}}$ [27,48] of $\mathbf{A} \in \mathbb{H}^{n\times n}$ presents the distinctive solution of

$$\mathbf{X} = \mathbf{XAX}, \qquad \mathcal{C}_r(\mathbf{X}) = \mathcal{C}_r(\mathbf{A}^{\mathrm{D}}) = \mathcal{C}_r(\mathbf{X}^{*}).$$

By [9, Theorem 2.3], recall that $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\mathrm{D}}\mathbf{A}^{\ell}\left(\mathbf{A}^{\ell}\right)^{\dagger}$, $\ell \geq \mathrm{Ind}(\mathbf{A})$. When $\mathrm{Ind}(\mathbf{A}) \leq 1$, $\mathbf{A}^{\textcircled{\dagger}} = \mathbf{A}^{\textcircled{\#}}$ is the core inverse of $\mathbf{A}$ [2].

The dual core-EP inverse $\mathbf{A}_{\textcircled{\dagger}}$ of $\mathbf{A} \in \mathbb{H}^{n\times n}$ is the unique solution to

$$\mathbf{X} = \mathbf{XAX}, \qquad \mathcal{R}_l(\mathbf{X}) = \mathcal{R}_l(\mathbf{A}^{\mathrm{D}}) = \mathcal{R}_l(\mathbf{X}^{*}),$$

and it can be given by $\mathbf{A}_{\textcircled{\dagger}} = \left(\mathbf{A}^{\ell}\right)^{\dagger}\mathbf{A}^{\ell}\mathbf{A}^{\mathrm{D}}$, $\ell \geq \mathrm{Ind}(\mathbf{A})$. Many interesting results related to the core-EP inverse are available in [6,7,9,30,34–36,38,39,45,47,49–51,56,58,59,61].

Since the quaternion core-EP inverse $\mathbf{A}^{\textcircled{\dagger}}$ is related to the right space $\mathcal{C}_r(\mathbf{A})$ of $\mathbf{A} \in \mathbb{H}^{n\times n}$ while the quaternion dual core-EP inverse $\mathbf{A}_{\textcircled{\dagger}}$ is related to the left space $\mathcal{R}_l(\mathbf{A})$, it is suggested to use notations as the right and left core-EP inverses, respectively, for these generalized inverses.

Extending results related to the core-EP inverse from [48], the right and left core-EP inverses are characterized in terms of some restricted equations in [28].

Lemma 1.1. [28, Lemma 6] *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *satisfy* $\mathrm{Ind}(\mathbf{A}) = k$. *Then* $\mathbf{X} \in \mathbb{H}^{n\times n}$ *is the right core-EP inverse of* $\mathbf{A}$ *if and only if*

$$\mathbf{XA}^{k+1} = \mathbf{A}^{k},\ \mathbf{AX}^{2} = \mathbf{X},\ (\mathbf{AX})^{*} = \mathbf{AX} \ \textit{and} \ \mathcal{C}_r(\mathbf{X}) \subseteq \mathcal{C}_r(\mathbf{A}^{k}).$$

Lemma 1.2. [28, Lemma 8] *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *and assume* $\mathrm{Ind}(\mathbf{A}) = k$. *The left core-EP pseudoinverse* $\mathbf{X} \in \mathbb{H}^{n\times n}$ *of* $\mathbf{A}$ *is defined as the solution to*

$$\mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^{k},\ \mathbf{X}^{2}\mathbf{A} = \mathbf{X},\ (\mathbf{XA})^{*} = \mathbf{XA} \ \textit{and} \ \mathcal{R}_l(\mathbf{X}) \subseteq \mathcal{R}_l(\mathbf{A}^{k}).$$

In order to improve presentation, the notation $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{K}_{m,n,k}$ will be used to denote the requirements

$$\mathbf{A} \in \mathbb{K}^{m\times n}, \mathbf{W} \in \mathbb{K}^{n\times m},\ k = \max\{\mathrm{Ind}(\mathbf{WA}), \mathrm{Ind}(\mathbf{AW})\}$$

over a selected algebraic structure $\mathbb{K}$.

The concept of the $\mathbf{W}$-weighted core-EP inverse in complex matrix case was introduced by Ferreyra *et al.* [8]. This notion is extended to the domain of quaternion matrices in [28].

Definition 1.1. *Let* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$. *The right* $\mathbf{W}$*-weighted core-EP (R-*$\mathbf{W}$*-CEP) pseudoinverse of* $\mathbf{A}$ *uniquely solves the restricted matrix equation*

$$\mathbf{WAWX} = (\mathbf{WA})^{k}\left[(\mathbf{WA})^{k}\right]^{\dagger},\ \ \mathcal{C}_r(\mathbf{X}) \subseteq \mathcal{C}_r\left((\mathbf{AW})^{k}\right).$$

Such $\mathbf{X}$ *is marked with* $\mathbf{A}^{\textcircled{\dagger},W_r}$.

According to [8], the R-**W**-CEP inverse over the quaternion skew field was determined in [28, Theorem 1] as follows.

Lemma 1.3. [28, Theorem 1] *Let* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$ *and* $\mathbf{X} \in \mathbb{H}^{m\times n}$. *The next claims are mutually equivalent:*

(i) $\mathbf{X} = \mathbf{A}^{\oplus, W_r}$;

(ii) $\mathbf{X}\mathbf{W}\left(\mathbf{A}\mathbf{W}\right)^{k+1} = \left(\mathbf{A}\mathbf{W}\right)^{k}, \ \mathbf{A}\mathbf{W}\mathbf{X}\mathbf{W}\mathbf{X} = \mathbf{X}, \quad \left(\mathbf{W}\mathbf{A}\mathbf{W}\mathbf{X}\right)^{*} = \mathbf{W}\mathbf{A}\mathbf{W}\mathbf{X}$;

(iii) $\mathbf{X} = \mathbf{A}\left[(\mathbf{W}\mathbf{A})^{\oplus}\right]^2$.

The left **W**-weighted core-EP (L-**W**-CEP) inverse was introduced in [28].

Definition 1.2. *Suppose that* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$. *The L-**W**-CEP pseudoinverse of* **A** *uniquely solves the restricted matrix equation*

$$\mathbf{X}\mathbf{W}\mathbf{A}\mathbf{W} = \left[(\mathbf{A}\mathbf{W})^{k}\right]^{\dagger}(\mathbf{A}\mathbf{W})^{k}, \ \ \mathcal{R}_l(\mathbf{X}) \subseteq \mathcal{R}_l\left((\mathbf{W}\mathbf{A})^{k}\right).$$

Such **X** *is marked with* $\mathbf{A}^{\oplus, W_l}$.

Lemma 1.4. [28, Theorem 2] *Let* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$ *and* $\mathbf{X} \in \mathbb{H}^{m\times n}$. *The subsequent claims are mutually equivalent:*

(i) $\mathbf{X} = \mathbf{A}^{\oplus, W_l}$;

(ii) **X** *uniquely solves the system*

$$(\mathbf{W}\mathbf{A})^{k+1}\mathbf{W}\mathbf{X} = (\mathbf{W}\mathbf{A})^{k}, \ \mathbf{X}\mathbf{W}\mathbf{X}\mathbf{W}\mathbf{A} = \mathbf{X}, \quad (\mathbf{X}\mathbf{W}\mathbf{A}\mathbf{W})^{*} = \mathbf{X}\mathbf{W}\mathbf{A}\mathbf{W};$$

(iii) $\mathbf{X} = \left[(\mathbf{A}\mathbf{W})_{\oplus}\right]^2\mathbf{A}$.

Many significant results related to **W**-weighted core-EP inverses are available in the recent literature. In [3, 10, 40, 42, 43], various characterizations of the **W**-weighted core-EP inverse and partial orders based on the **W**-weighted core-EP inverse were developed. Perturbation estimations which are related to the **W**-weighted core-EP inverse were established in [37, 41]. Applying the **W**-weighted core-EP inverses, solvability of some constrained approximation problems were given in [45] for complex matrices and in [31] for quaternion matrices.

In [4], the notions of the MPCEP inverse and ∗CEPMP inverse were introduced in terms of the (dual) core-EP inverse and the MP inverse for Hilbert space operators. Based on quaternion core-EP inverses, these definitions were extended to quaternion matrices [29].

We investigate generalizations of the core-EP inverses from the set of complex matrices to the more general quaternion skew field domain. The motivation is caused by significantly increasing interest in quaternion matrix equations in the last years that is based on their wide applications in various fields, among them, robotic manipulation [55], fluid mechanics [11, 12], quantum mechanics [1, 18, 33], signal processing [53, 54], color image processing [16, 17, 32], and so on.

Recently, Stojanović and Mosić [52] defined three novel weighted generalized inverses for bounded linear operators between Hilbert spaces which can be considered for complex matrices as in Definition 1.3.

Definition 1.3. *The subsequent statements hold for* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{C}_{m,n,k}$.

(a) *The* $\mathbf{W}$*-weighted MP-core-EP (*$\mathbf{W}$*-MPCEP) inverse of* $\mathbf{A}$ *is established as*

$$\mathbf{A}^{\dagger,\oplus,W} = \mathbf{A}^{\dagger}\mathbf{AWA}^{\oplus,W_r}\mathbf{W}.$$

(b) *The* $\mathbf{W}$*-weighted core-EP-MP (*$\mathbf{W}$*-CEPMP) inverse of* $\mathbf{A}$ *is established as*

$$\mathbf{A}^{\oplus,\dagger,W} = \mathbf{WA}^{\oplus,W_r}\mathbf{WAA}^{\dagger}.$$

(c) *The* $\mathbf{W}$*-weighted MP-core-EP-MP (*$\mathbf{W}$*-MPCEPMP) inverse of* $\mathbf{A}$ *is established as*

$$\mathbf{A}^{\dagger,\oplus,\dagger,W} = \mathbf{A}^{\dagger}\mathbf{AWA}^{\oplus,W_r}\mathbf{WAA}^{\dagger}.$$

It is worth mentioning that the $\mathbf{W}$-MPCEP inverse presents a generalization of the MPCEP inverse, which was proposed in [4]. Some applications of MPCEP inverse were proved in [29, 44] for solving certain types of linear equations.

Determinantal representations ($\mathfrak{D}$-representations) of the complex core inverse as well as its various generalizations were proposed in [25, 26, 48]. In the quaternion case, we meet the problem of defining a determinant of a quaternion matrix. Recently, this problem was solved thanks to the developed theory of row-determinants ($\mathfrak{R}$-determinants) and column-determinants ($\mathfrak{C}$-determinants), introduced in [19, 20]. $\mathfrak{D}$-representations of generalized inverses are expressed in terms of the $\mathfrak{R}$- and $\mathfrak{C}$-determinants [21–24].

The notation $\mathbf{W}$-MP$\leftrightarrows$CEP inverses will be used as common for $\mathbf{W}$-MPCEP, $\mathbf{W}$-CEPMP and $\mathbf{W}$-MPCEPMP inverses.

Motivated by the notion of $\mathbf{W}$-MP$\leftrightarrows$CEP inverses of complex matrices, we define the quaternion left and right $\mathbf{W}$-MP$\leftrightarrows$CEP inverses in order to continue this topic on quaternion matrices. The quaternion left and right $\mathbf{W}$-MP$\leftrightarrows$CEP inverses will be denoted by L-$\mathbf{W}$-MP$\leftrightarrows$CEP and R-$\mathbf{W}$-MP$\leftrightarrows$CEP inverses, respectively. The new presented weighted generalized inverses are proposed to solve certain systems of linear quaternion equations. Precisely, the next investigation streams will be considered.

(1) Based on adequate combinations of the MP inverse with the left and right $\mathbf{W}$-weighted core-EP pseudoinverses, we introduce the L-$\mathbf{W}$-MP$\leftrightarrows$CEP and R-$\mathbf{W}$-MP$\leftrightarrows$CEP inverses for quaternion matrices.

(2) New expressions and characterizations are proposed for the introduced L-$\mathbf{W}$-MP$\leftrightarrows$CEP and R-$\mathbf{W}$-MP$\leftrightarrows$CEP inverses.

(3) $\mathfrak{D}$-representations for the L-$\mathbf{W}$-MP$\leftrightarrows$CEP and R-$\mathbf{W}$-MP$\leftrightarrows$CEP inverses are presented.

(4) An expository example is stated for the illustration of derived results.

The content of our article follows. Section 2 contains definitions for the L-$\mathbf{W}$-MP$\leftrightarrows$CEP and R-$\mathbf{W}$-MP$\leftrightarrows$CEP inverses of quaternion matrices as well as their representations and characterizations. In Section 3, $\mathfrak{D}$-representations of L-$\mathbf{W}$-MP$\leftrightarrows$CEP and R-$\mathbf{W}$-MP$\leftrightarrows$CEP inverses are developed. A numerical example about derived results is presented in Section 4.

2 Characterizations and representations of quaternion W-MPCEP, W-CEPMP and W-MPCEPMP inverses

According to the proposed concept, we consider both the left and right $\mathbf{W}$-core-EP pseudoinverses over quaternion skew fields to define the L-$\mathbf{W}$-MP⇆CEP and R-$\mathbf{W}$-MP⇆CEP inverses.

Definition 2.1. *Suppose that* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$.

(a) *The R-$\mathbf{W}$-MPCEP and L-$\mathbf{W}$-MPCEP inverses of $\mathbf{A}$ are defined, respectively, as*

$$\mathbf{A}^{\dagger,\textcircled{\dagger},W_r} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_r}\mathbf{W} \quad \textit{and} \quad \mathbf{A}^{\dagger,\textcircled{\dagger},W_l} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_l}\mathbf{W}.$$

(b) *The R-$\mathbf{W}$-CEPMP and L-$\mathbf{W}$-CEPMP inverses of $\mathbf{A}$ are defined as*

$$\mathbf{A}^{\textcircled{\dagger},\dagger,W_r} = \mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_r}\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger} \quad \textit{and} \quad \mathbf{A}^{\textcircled{\dagger},\dagger,W_l} = \mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_l}\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger}.$$

(c) *The R-$\mathbf{W}$-MPCEPMP and L-$\mathbf{W}$-MPCEPMP inverses of $\mathbf{A}$ are defined as*

$$\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_r} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_r}\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger} \quad \textit{and} \quad \mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_l} = \mathbf{A}^{\dagger}\mathbf{A}\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_l}\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger}.$$

Theorem 2.1. *The subsequent representations arise for $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$, .*

(a) *The R-$\mathbf{W}$-MPCEP and L-$\mathbf{W}$-MPCEP inverses of A can be expressed as*

$$\mathbf{A}^{\dagger,\textcircled{\dagger},W_r} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}, \tag{2.1}$$
$$\mathbf{A}^{\dagger,\textcircled{\dagger},W_l} = \mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}. \tag{2.2}$$

(b) *The R-$\mathbf{W}$-CEPMP and L-$\mathbf{W}$-CEPMP inverses of A can be expressed as*

$$\mathbf{A}^{\textcircled{\dagger},\dagger,W_r} =(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{P}_A, \tag{2.3}$$
$$\mathbf{A}^{\textcircled{\dagger},\dagger,W_l} =\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{P}_A. \tag{2.4}$$

(c) *The R-$\mathbf{W}$-MPCEPMP and L-$\mathbf{W}$-MPCEPMP inverses of A can be expressed as*

$$\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_r} =\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{P}_A, \tag{2.5}$$
$$\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_l} =\mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{P}_A. \tag{2.6}$$

Proof. For the right ones, we have the next. Since $\mathbf{A}^{\textcircled{\dagger},W_r} = \mathbf{A}\left[(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\right]^2$ and due to Lemma 1.1, it follows $\mathbf{V}\left[\mathbf{V}^{\textcircled{\dagger}}\right]^2 = \mathbf{V}^{\textcircled{\dagger}}$. Then, in view of Lemma 1.3, it follows

$$\begin{aligned}
\mathbf{A}^{\dagger,\textcircled{\dagger},W_r} &=\mathbf{A}^{\dagger}\mathbf{A}\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_r}\mathbf{W} = \mathbf{A}^{\dagger}\mathbf{A}(\mathbf{W}\mathbf{A})\left[(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\right]^2\mathbf{W} = \mathbf{Q}_{\mathbf{A}}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W},\\
\mathbf{A}^{\textcircled{\dagger},\dagger,W_r} &=\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_r}\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger} = (\mathbf{W}\mathbf{A})\left[(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\right]^2\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger} = (\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{P}_A,\\
\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_r} &=\mathbf{A}^{\dagger}\mathbf{A}\mathbf{W}\mathbf{A}^{\textcircled{\dagger},W_r}\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}\mathbf{A}(\mathbf{W}\mathbf{A})\left[(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\right]^2\mathbf{W}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{P}_A.
\end{aligned}$$

For the left ones, it follows. Since $\mathbf{A}^{\oplus,W_l} = \left[(\mathbf{AW})_{\oplus}\right]^2 \mathbf{A}$ and due to Lemma 1.2, it follows $\left[\mathbf{V}_{\oplus}\right]^2 \mathbf{V} = \mathbf{V}_{\oplus}$, which implies in conjunction with Lemma 1.4

$$\begin{aligned}
\mathbf{A}^{\dagger,\oplus,W_l} =& \mathbf{A}^\dagger\mathbf{AWA}^{\oplus,W_l}\mathbf{W} = \mathbf{A}^\dagger\mathbf{AW}\left[(\mathbf{AW})_{\oplus}\right]^2\mathbf{AW} = \mathbf{Q}_A\mathbf{W}\left(\mathbf{AW}\right)_{\oplus},\\
\mathbf{A}^{\oplus,\dagger,W_l} =& \mathbf{WA}^{\oplus,W_l}\mathbf{WAA}^\dagger = \mathbf{W}\left[(\mathbf{AW})_{\oplus}\right]^2(\mathbf{AW})\mathbf{AA}^\dagger = \mathbf{W}\left(\mathbf{AW}\right)_{\oplus}\mathbf{P}_A,\\
\mathbf{A}^{\dagger,\oplus,\dagger,W_l} =& \mathbf{A}^\dagger\mathbf{AWA}^{\oplus,W_l}\mathbf{WAA}^\dagger = \mathbf{A}^\dagger\mathbf{AW}\left[(\mathbf{AW})_{\oplus}\right]^2(\mathbf{AW})\mathbf{AA}^\dagger = \mathbf{Q}_A\mathbf{W}\left(\mathbf{AW}\right)_{\oplus}\mathbf{P}_A.
\end{aligned}$$

The proof is finished. □

As a consequence of Theorem 2.1, we obtain new representations of L-**W**-MP⇆CEP and R-**W**-MP⇆CEP inverses.

Corollary 2.1. *Let* $\{\mathbf{A},\mathbf{W}\} \in \mathbb{H}_{m,n,k}$*, and* $\ell \geq k$*.*

(a) *Then the R-**W**-MPCEP and L-**W**-MPCEP inverses of* A *can be expressed as*

$$\begin{aligned}
\mathbf{A}^{\dagger,\oplus,W_r} &= \mathbf{Q}_A(\mathbf{WA})^{\mathrm{D}}\mathbf{P}_{(WA)^\ell}\mathbf{W} = \mathbf{Q}_A(\mathbf{WA})^\ell\left((\mathbf{WA})^{\ell+1}\right)^\dagger\mathbf{W},\\
\mathbf{A}^{\dagger,\oplus,W_l} &= \mathbf{Q}_A\mathbf{W}\mathbf{Q}_{(AW)^\ell}(\mathbf{AW})^{\mathrm{D}} = \mathbf{Q}_A\mathbf{W}\left((\mathbf{AW})^{\ell+1}\right)^\dagger(\mathbf{AW})^\ell.
\end{aligned}$$

(b) *Then the R-**W**-CEPMP and L-**W**-CEPMP inverses of* $\mathbf{A}$ *can be expressed as*

$$\begin{aligned}
\mathbf{A}^{\oplus,\dagger,W_r} =& (\mathbf{WA})^{\mathrm{D}}\mathbf{P}_{(WA)^\ell}\mathbf{WP}_A = (\mathbf{WA})^\ell\left((\mathbf{WA})^{\ell+1}\right)^\dagger\mathbf{WP}_A,\\
\mathbf{A}^{\oplus,\dagger,W_l} =& \mathbf{WQ}_{(AW)^l}(\mathbf{AW})^{\mathrm{D}}\mathbf{P}_A = \mathbf{W}\left((\mathbf{AW})^{\ell+1}\right)^\dagger(\mathbf{AW})^\ell\mathbf{P}_A.
\end{aligned}$$

(c) *Then R-**W**-MPCEPMP and L-**W**-MPCEPMP inverses of* $\mathbf{A}$ *can be expressed as*

$$\begin{aligned}
\mathbf{A}^{\dagger,\oplus,\dagger,W_r} =& \mathbf{Q}_A(\mathbf{WA})^{\mathrm{D}}\mathbf{P}_{(WA)^\ell}\mathbf{WP}_A = \mathbf{Q}_A(\mathbf{WA})^\ell\left((\mathbf{WA})^{\ell+1}\right)^\dagger\mathbf{WP}_A,\\
\mathbf{A}^{\dagger,\oplus,\dagger,W_l} =& \mathbf{Q}_A\mathbf{WQ}_{(AW)^\ell}(\mathbf{AW})^{\mathrm{D}}\mathbf{P}_A = \mathbf{Q}_A\mathbf{W}\left((\mathbf{AW})^{\ell+1}\right)^\dagger(\mathbf{AW})^\ell\mathbf{P}_A.
\end{aligned}$$

Proof. It follows by Theorem 2.1,

$$\begin{aligned}
(\mathbf{WA})^{\oplus} &= (\mathbf{WA})^{\mathrm{D}}(\mathbf{WA})^\ell\left((\mathbf{WA})^\ell\right)^\dagger = (\mathbf{WA})^{\mathrm{D}}\mathbf{P}_{(\mathbf{WA})^\ell} = (\mathbf{WA})^{\mathrm{D}}\mathbf{P}_{(WA)^{\ell+1}}\\
&= (\mathbf{WA})^{\mathrm{D}}(\mathbf{WA})^{\ell+1}\left((\mathbf{WA})^{\ell+1}\right)^\dagger = (\mathbf{WA})^\ell\left((\mathbf{WA})^{\ell+1}\right)^\dagger
\end{aligned}$$

and

$$\begin{aligned}
(\mathbf{AW})_{\oplus} &= \left((\mathbf{AW})^\ell\right)^\dagger(\mathbf{AW})^\ell(\mathbf{AW})^{\mathrm{D}} = \mathbf{Q}_{(\mathbf{AW})^\ell}(\mathbf{AW})^{\mathrm{D}} = \mathbf{Q}_{(\mathbf{AW})^{\ell+1}}(\mathbf{AW})^{\mathrm{D}}\\
&= \left((\mathbf{AW})^{\ell+1}\right)^\dagger(\mathbf{AW})^{\ell+1}(\mathbf{AW})^{\mathrm{D}} = \left((\mathbf{AW})^{\ell+1}\right)^\dagger(\mathbf{AW})^\ell.
\end{aligned}$$

□

Theorem 2.2 gives the characteristic equations for introduced L-$\mathbf{W}$-MP⇆CEP and R-$\mathbf{W}$-MP⇆CEP inverses. These results generalize characterizations given in [52].

Theorem 2.2. *Let* $\{\mathbf{A},\mathbf{W}\}\in\mathbb{H}_{m,n,k}$.

(a) *Then* $\mathbf{B}_1=\mathbf{A}^{\dagger,\textcircled{\dagger},W_r}$ *are* $\mathbf{B}_2=\mathbf{A}^{\dagger,\textcircled{\dagger},W_l}$*, respectively, are the unique solutions to the systems of equations*

$$\mathbf{B}_1\mathbf{A}\mathbf{B}_1=\mathbf{B}_1,\quad \mathbf{A}\mathbf{B}_1=\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W},\quad \mathbf{B}_1\mathbf{A}=\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}, \tag{2.7}$$

$$\mathbf{B}_2\mathbf{A}\mathbf{B}_2=\mathbf{B}_2,\quad \mathbf{A}\mathbf{B}_2=\mathbf{A}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}},\quad \mathbf{B}_2\mathbf{A}=\mathbf{Q}_{\mathbf{A}}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{A}. \tag{2.8}$$

(b) *Then* $\mathbf{C}_1=\mathbf{A}^{\textcircled{\dagger},\dagger,W_r}$ *and* $\mathbf{C}_2=\mathbf{A}^{\textcircled{\dagger},\dagger,W_l}$*, respectively, are the unique solutions to the systems*

$$\mathbf{C}_1\mathbf{A}\mathbf{C}_1=\mathbf{C}_1,\quad \mathbf{A}\mathbf{C}_1=\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{P}_A,\quad \mathbf{C}_1\mathbf{A}=(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A},$$

$$\mathbf{C}_2\mathbf{A}\mathbf{C}_2=\mathbf{C}_2,\quad \mathbf{A}\mathbf{C}_2=\mathbf{A}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{P}_A,\quad \mathbf{C}_2\mathbf{A}=\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{A}.$$

(c) *Then* $\mathbf{D}_1=\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_r}$ *and* $\mathbf{D}_2=\mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_l}$*, respectively, are the unique solutions to the systems*

$$\mathbf{D}_1\mathbf{A}\mathbf{D}_1=\mathbf{D}_1,\quad \mathbf{A}\mathbf{D}_1=\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{P}_A,\quad \mathbf{D}_1\mathbf{A}=\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A},$$

$$\mathbf{D}_2\mathbf{A}\mathbf{D}_2=\mathbf{D}_2,\quad \mathbf{A}\mathbf{D}_2=\mathbf{A}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{P}_A,\quad \mathbf{D}_2\mathbf{A}=\mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{A}.$$

Proof. We firstly verify the condition (2.7). Since $\mathbf{B}_1=\mathbf{A}^{\dagger,\textcircled{\dagger},W_r}=\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}$, then, by Lemma 1.1, we have

$$\begin{aligned}\mathbf{B}_1\mathbf{A}\mathbf{B}_1=&\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}(\mathbf{A}\mathbf{Q}_A)(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}=\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}(\mathbf{W}\mathbf{A})(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}=\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}=\mathbf{B}_1,\\ \mathbf{A}\mathbf{B}_1=&\mathbf{A}\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}=\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W},\\ \mathbf{B}_1\mathbf{A}=&\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}.\end{aligned}$$

Now, consider the condition (2.8) and $\mathbf{B}_2=\mathbf{A}^{\dagger,\textcircled{\dagger},W_l}=\mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}$. Then the following holds by Lemma 1.2

$$\begin{aligned}\mathbf{B}_2\mathbf{A}\mathbf{B}_2=&\mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}(\mathbf{A}\mathbf{Q}_{\mathbf{A}})\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}=\mathbf{Q}_{\mathbf{A}}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}(\mathbf{A}\mathbf{W})(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}=\mathbf{Q}_{\mathbf{A}}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}=\mathbf{B_2},\\ \mathbf{A}\mathbf{B}_2=&\mathbf{A}\mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}=\mathbf{A}\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}},\\ \mathbf{B}_2\mathbf{A}=&\mathbf{Q}_A\mathbf{W}(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}}\mathbf{A}.\end{aligned}$$

The uniqueness of $\mathbf{B}_1$ and $\mathbf{B}_2$ as the solutions of the systems (2.7) and (2.8), respectively, can be proved following the principles from [52].

The parts (b) and (c) are verified in the same way. □

The next characterizations of L-$\mathbf{W}$-MP⇆CEP and R-$\mathbf{W}$-MP⇆CEP inverses are derived on the basis of Theorem 2.2.

Corollary 2.2. *Let* $\{\mathbf{A},\mathbf{W}\}\in\mathbb{H}_{m,n,k}$ *and* $\ell\geq k$.

(a) *Then* $\mathbf{B}_1 = \mathbf{A}^{\dagger,\textcircled{\dagger},W_r}$ *are* $\mathbf{B}_2 = \mathbf{A}^{\dagger,\textcircled{\dagger},W_l}$*, respectively, are the unique solutions to the matrix systems*

$$\mathbf{B}_1\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1, \quad \mathbf{A}\mathbf{B}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger \mathbf{W}, \quad \mathbf{B}_1\mathbf{A} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger \mathbf{W}\mathbf{A},$$

$$\mathbf{B}_2\mathbf{A}\mathbf{B}_2 = \mathbf{B}_2, \quad \mathbf{A}\mathbf{B}_2 = \mathbf{A}\mathbf{W} \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell, \quad \mathbf{B}_2\mathbf{A} = \mathbf{Q}_A\mathbf{W} \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell\mathbf{A}.$$

(b) *Then* $\mathbf{C}_1 = \mathbf{A}^{\textcircled{\dagger},\dagger,W_r}$ *and* $\mathbf{C}_2 = \mathbf{A}^{\textcircled{\dagger},\dagger,W_l}$*, respectively, are the unique solutions to the systems*

$$\mathbf{C}_1\mathbf{A}\mathbf{C}_1 = \mathbf{C}_1, \quad \mathbf{A}\mathbf{C}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger \mathbf{W}\mathbf{P}_A, \quad \mathbf{C}_1\mathbf{A} = (\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger \mathbf{W}\mathbf{A},$$

$$\mathbf{C}_2\mathbf{A}\mathbf{C}_2 = \mathbf{C}_2, \quad \mathbf{A}\mathbf{C}_2 = \mathbf{A}\mathbf{W} \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell\mathbf{P}_A, \quad \mathbf{C}_2\mathbf{A} = \mathbf{W} \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell\mathbf{A}.$$

(c) *Then* $\mathbf{D}_1 = \mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_r}$ *and* $\mathbf{D}_2 = \mathbf{A}^{\dagger,\textcircled{\dagger},\dagger,W_l}$*, respectively, are the unique solutions to the systems*

$$\mathbf{D}_1\mathbf{A}\mathbf{D}_1 = \mathbf{D}_1, \quad \mathbf{A}\mathbf{D}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger \mathbf{W}\mathbf{P}_A, \quad \mathbf{D}_1\mathbf{A} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger \mathbf{W}\mathbf{A},$$

$$\mathbf{D}_2\mathbf{A}\mathbf{D}_2 = \mathbf{D}_2, \quad \mathbf{A}\mathbf{D}_2 = \mathbf{A}\mathbf{W} \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell\mathbf{P}_A, \quad \mathbf{D}_2\mathbf{A} = \mathbf{Q}_A\mathbf{W} \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell\mathbf{A}.$$

Proof. Applying Theorem 2.2, $(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}} = (\mathbf{W}\mathbf{A})^\ell \left((\mathbf{W}\mathbf{A})^{\ell+1}\right)^\dagger$ and $(\mathbf{A}\mathbf{W})_{\textcircled{\dagger}} = \left((\mathbf{A}\mathbf{W})^{\ell+1}\right)^\dagger (\mathbf{A}\mathbf{W})^\ell$, we complete this proof. □

According to [46, Theorem 2.3], the next characterizations for a given matrix to become the R-**W**-MPCEP inverse are obtained.

Corollary 2.3. *Let* $\{\mathbf{A}, \mathbf{W}\} \in \mathbb{H}_{m,n,k}$ *and* $\ell \geq k$*. For* $\mathbf{B}_1 \in \mathbb{H}^{n\times m}$*, the following statements are equivalent:*

(i) $\mathbf{B}_1 = \mathbf{A}^{\dagger,\textcircled{\dagger},W_r}$*;*

(ii) $\mathbf{B}_1\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1$*,* $\mathbf{A}\mathbf{B}_1\mathbf{A} = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}$*,* $\mathbf{A}\mathbf{B}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}$ *and* $\mathbf{B}_1\mathbf{A} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}$*;*

(iii) $\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1$ *and* $\mathbf{A}\mathbf{B}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}$*;*

(iv) $\mathbf{B}_1\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W} = \mathbf{B}_1$ *and* $\mathbf{B}_1\mathbf{A} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}$*;*

(v) $\mathbf{B}_1\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1$*,* $\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A} = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}$*,* $\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}$ *and* $\mathbf{B}_1\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}$*;*

(vi) $\mathbf{B}_1\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1$*,* $\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}$ *and* $\mathbf{B}_1\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A} = \mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}$*;*

(vii) $\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1$ *and* $\mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{A}(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}$*;*

(viii) $\mathbf{Q}_A(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{B}_1$ *and* $\mathbf{A}\left[(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\right]^2\mathbf{W}\mathbf{A}\mathbf{B}_1 = \mathbf{A}\left[(\mathbf{W}\mathbf{A})^{\textcircled{\dagger}}\right]^2\mathbf{W}$*;*

(ix) $\mathbf{B}_1\mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{W} = \mathbf{B}_1$ *and* $\mathbf{B}_1\mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{WA} = \mathbf{Q}_A(\mathbf{WA})^{\oplus}\mathbf{WA}$*;*

(x) $\mathbf{Q}_A\mathbf{B}_1 = \mathbf{B}_1$ *and* $\mathbf{AB}_1 = \mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{W}$*;*

(xi) $\mathbf{Q}_A\mathbf{B}_1 = \mathbf{B}_1$ *and* $\mathbf{A}^*\mathbf{AB}_1 = \mathbf{A}^*\mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{W}$*;*

(xii) $\mathbf{A}\left[(\mathbf{WA})^{\oplus}\right]^2\mathbf{WAB}_1\mathbf{A} = \mathbf{A}\left[(\mathbf{WA})^{\oplus}\right]^2\mathbf{WA}$ *and* $\mathbf{Q}_A(\mathbf{WA})^{\oplus}\mathbf{WAB}_1 = \mathbf{B}_1$*;*

(xiii) $\mathbf{A}\left[(\mathbf{WA})^{\oplus}\right]^2\mathbf{WAB}_1\mathbf{AA}^* = \mathbf{A}\left[(\mathbf{WA})^{\oplus}\right]^2\mathbf{WAA}^*$ *and* $\mathbf{Q}_A(\mathbf{WA})^{\oplus}\mathbf{WAB}_1 = \mathbf{B}_1$*;*

(xiv) $\mathbf{B}_1\mathbf{A}(\mathbf{WA})^{\oplus} = \mathbf{Q}_A(\mathbf{WA})^{\oplus}$ *and* $\mathbf{B}_1\mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{W} = \mathbf{B}_1$*;*

(xv) $\mathbf{AB}_1\mathbf{A}(\mathbf{WA})^{\oplus} = \mathbf{A}(\mathbf{WA})^{\oplus}$ *and* $\mathbf{Q}_A\mathbf{B}_1\mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{W} = \mathbf{B}_1$*;*

(xvi) $\mathbf{A}^*\mathbf{AB}_1\mathbf{A}(\mathbf{WA})^{\oplus} = \mathbf{A}^*\mathbf{A}(\mathbf{WA})^{\oplus}$ *and* $\mathbf{Q}_A\mathbf{B}_1\mathbf{A}(\mathbf{WA})^{\oplus}\mathbf{W} = \mathbf{B}_1$.

Replacing $(\mathbf{WA})^{\oplus}$ in Corollary 2.3 with some of the expressions $(\mathbf{WA})^{\oplus} = (\mathbf{WA})^{\mathrm{D}}\mathbf{P}_{(WA)^\ell} = (\mathbf{WA})^\ell\left((\mathbf{WA})^{\ell+1}\right)^\dagger$, it is possible to obtain further characterizations for the right **W**-MPCEP inverse. In an analogous way, we can present characterizations for the L-**W**-MPCEP inverse and R- or L- **W**-CEPMP and **W**-MPCEPMP inverses.

3 Determinantal representations of W-MPCEP, W-CEPMP and W-MPCEPMP inverses

Based on $\mathfrak{D}$-representations of the MP inverse, $\mathfrak{D}$-representations of the initiated projections $\mathbf{Q}_A = \mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{P}_A = \mathbf{AA}^\dagger$ are obtained in [21].

Lemma 3.1. [21] *For* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$*, the* $\mathfrak{D}$*-representations of* $\mathbf{Q}_A = \left(q_{ij}^A\right)_{n\times n}$ *and* $\mathbf{P}_A = \left(p_{ij}^A\right)_{m\times m}$ *are elementwise given as*

$$q_{ij}^A = \frac{\sum\limits_{\psi\in J_{r,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.j}\right)\right)_\psi^\psi}{\sum\limits_{\psi\in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_\psi^\psi} = \frac{\sum\limits_{\mu\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^*\mathbf{A})_{.j}\left(\dot{\mathbf{a}}_{.i}\right)\right)_\mu^\mu}{\sum\limits_{\mu\in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_\mu^\mu}, \tag{3.1}$$

$$p_{ij}^A = \frac{\sum\limits_{\mu\in I_{r,m}\{j\}} \operatorname{rdet}_j((\mathbf{AA}^*)_{j.}(\ddot{\mathbf{a}}_{i.}))_\mu^\mu}{\sum\limits_{\mu\in I_{r,m}} |\mathbf{AA}^*|_\mu^\mu} = \frac{\sum\limits_{\psi\in J_{r,m}\{i\}} \operatorname{cdet}_i((\mathbf{AA}^*)_{.i}(\ddot{\mathbf{a}}_{.j}))_\psi^\psi}{\sum\limits_{\psi\in J_{r,m}} |\mathbf{AA}^*|_\psi^\psi}, \tag{3.2}$$

in which $\dot{\mathbf{a}}_{.i}$, $\dot{\mathbf{a}}_{.j}$, *(resp.* $\ddot{\mathbf{a}}_{i.}$, $\ddot{\mathbf{a}}_{.j}$*) stand for ith row and jth column of* $\mathbf{A}^*\mathbf{A} \in \mathbb{H}^{n\times n}$ *(resp.* $\mathbf{AA}^* \in \mathbb{H}^{m\times m}$*).*

In [27], $\mathfrak{D}$-representations of both left and right core-EP inverses are given.

Lemma 3.2. [27] *Assume that* $\mathbf{A}\in\mathbb{H}^{n\times n}$ *satisfies* $\mathrm{Ind}(\mathbf{A})=k$ *and* $\mathrm{rank}(\mathbf{A}^k)=s$. *Under these conditions,* $\mathfrak{D}$*-representations of* $\mathbf{A}^{\textcircled{\dagger}}=\left(a_{ij}^{\textcircled{\dagger},r}\right)$ *and* $\mathbf{A}_{\textcircled{\dagger}}=\left(a_{ij}^{\textcircled{\dagger},l}\right)$ *are equal to*

$$a_{ij}^{\textcircled{\dagger},r}=\frac{\sum\limits_{\mu\in I_{s,n}\{j\}}\mathrm{rdet}_j\left(\mathbf{M}_{j.}(\hat{\mathbf{a}}_{i.})\right)_\mu^\mu}{\sum\limits_{\mu\in I_{s,n}}\left|\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*\right|_\mu^\mu}, \tag{3.3}$$

$$a_{ij}^{\textcircled{\dagger},l}=\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\mathrm{cdet}_i\left(\mathbf{N}_{.i}(\check{\mathbf{a}}_{.j})\right)_\psi^\psi}{\sum\limits_{\psi\in J_{s,n}}\left|\left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^{k+1}\right|_\psi^\psi}, \tag{3.4}$$

wherein $\mathbf{M}=\mathbf{A}^{k+1}\left(\mathbf{A}^{k+1}\right)^*$, $\mathbf{N}=\left(\mathbf{A}^{k+1}\right)^*\mathbf{A}^{k+1}$ *and* $\hat{\mathbf{a}}_{i.}$ *(resp.* $\check{\mathbf{a}}_{.j}$*) stands for the* i*th row of* $\hat{\mathbf{A}}=\mathbf{A}^k(\mathbf{A}^{k+1})^*$ *(resp.* j*th column of* $\check{\mathbf{A}}=(\mathbf{A}^{k+1})^*\mathbf{A}^k$*).*

Let $\mathbf{WA}=\mathbf{U}=(u_{ij})\in\mathbb{H}^{n\times n}$ and $\mathbf{AW}=\mathbf{V}=(v_{ij})\in\mathbb{H}^{m\times m}$. Theorem 3.1 develops the $\mathfrak{D}$-representations of the quaternion L-**W**-MP⇆CEP and R-**W**-MP⇆CEP inverses.

Theorem 3.1. *Assume that* $\mathbf{A}=(a_{ij})\in\mathbb{H}_s^{m\times n}$ *and* $\mathbf{W}=(w_{ij})\neq 0\in\mathbb{H}^{n\times m}$ *satisfy* $\{\mathbf{A},\mathbf{W}\}\in\mathbb{H}_{m,n,k}$ *as well as* $\mathrm{rank}(\mathbf{U})=s_1$, $\mathrm{rank}(\mathbf{V})=s_2$.

(1) *Then the R-*$\mathbf{W}$*-MPCEP inverse* $\mathbf{A}^{\dagger,\textcircled{\dagger},W_r}=\left(a_{ij}^{\dagger,\textcircled{\dagger},W_r}\right)$ *can be represented as*

$$a_{ij}^{\dagger,\textcircled{\dagger},W_r}=\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.j}^{(1)}\right)\right)_\psi^\psi}{\sum\limits_{\psi\in J_{s,n}}\left|\mathbf{A}^*\mathbf{A}\right|_\psi^\psi\sum\limits_{\mu\in I_{s_1,n}}\left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|_\mu^\mu}, \tag{3.5}$$

with $\mathbf{a}_{.j}^{(1)}$ *representing the* j*th column of* $\mathbf{A}_1=\mathbf{A}^*\mathbf{A}\mathbf{U}_1\mathbf{W}$. *Further,* $\mathbf{U}_1=(u_{ft}^{(1)})$ *fulfils*

$$u_{ft}^{(1)}=\sum_{\mu\in I_{s_1,n}\{t\}}\mathrm{rdet}_t\left(\left(\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right)_{t.}(\hat{\mathbf{u}}_{f.})\right)_\mu^\mu, \tag{3.6}$$

and $\hat{\mathbf{u}}_{f.}$ *stands for* f*th row from* $\hat{\mathbf{U}}=\mathbf{U}^k(\mathbf{U}^{k+1})^*$.

(2) *Then the L-*$\mathbf{W}$*-MPCEP pseudoinverse* $\mathbf{A}^{\dagger,\textcircled{\dagger},W_l}=\left(a_{ij}^{\dagger,\textcircled{\dagger},W_l}\right)$ *can be represented as*

$$a_{ij}^{\dagger,\textcircled{\dagger},W_l}=\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.j}^{(2)}\right)\right)_\psi^\psi}{\sum\limits_{\psi\in J_{s,n}}\left|\mathbf{A}^*\mathbf{A}\right|_\psi^\psi\sum\limits_{\mu\in I_{s_2,m}}\left|\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right|_\psi^\psi}, \tag{3.7}$$

where $\mathbf{a}_{.j}^{(2)}$ *is* j*th column of* $\mathbf{A}_2=\mathbf{A}^*\mathbf{AWV}_1$. *Further* $\mathbf{V}_1=(v_{fj}^{(1)})$ *satisfies*

$$v_{tj}^{(1)}=\sum_{\psi\in J_{s_2,m}\{t\}}\mathrm{cdet}_t\left(\left(\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right)_{.t}(\check{\mathbf{v}}_{.j})\right)_\psi^\psi, \tag{3.8}$$

such that $\check{\mathbf{v}}_{.j}$ *means* j*th column from* $\check{\mathbf{V}}=(\mathbf{V}^{k+1})^*\mathbf{V}^k$.

Proof. (1) Thanks to (2.1), we have

$$a_{ij}^{\dagger,\textcircled{\dagger},W_r}=\sum_{f=1}^{n}\sum_{t=1}^{n}q_{if}^{A}u_{ft}^{\textcircled{\dagger},r}w_{tj}.$$

Using the $\mathfrak{D}$-representation of $\mathbf{Q}_A=\mathbf{A}^\dagger\mathbf{A}=(q_{if}^{A})$ given by (3.1) and the $\mathfrak{D}$-representation of $\mathbf{U}^{\textcircled{\dagger}}=(u_{ft}^{\textcircled{\dagger},r})$ as in (3.3), we obtain

$$a_{ij}^{\dagger,\textcircled{\dagger},W_r}=\sum_{f=1}^{n}\sum_{t=1}^{n}\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)_{\psi}^{\psi}}{\sum\limits_{\psi\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_{\psi}^{\psi}}\frac{\sum\limits_{\mu\in I_{s_1,n}\{t\}}\operatorname{rdet}_t\left(\left(\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right)_{t.}\left(\hat{\mathbf{u}}_{f.}\right)\right)_{\mu}^{\mu}w_{tj}}{\sum\limits_{\mu\in I_{s_1,n}}\left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|_{\mu}^{\mu}},$$

where $\dot{\mathbf{a}}_{.f}$ is the fth column of $\mathbf{A}^*\mathbf{A}$ and $\hat{\mathbf{u}}_{f.}$ stands for the fth row of $\hat{\mathbf{U}}=\mathbf{U}^k(\mathbf{U}^{k+1})^*$.

Denote by

$$u_{ft}^{(1)}:=\sum_{\mu\in I_{s_1,n}\{t\}}\operatorname{rdet}_t\left(\left(\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right)_{t.}\left(\hat{\mathbf{u}}_{f.}\right)\right)_{\mu}^{\mu}$$

the (ft)th element of the matrix $\mathbf{U}_1=\left(u_{ft}^{(1)}\right)$ and observe $\mathbf{A}_1=\mathbf{A}^*\mathbf{A}\mathbf{U}_1\mathbf{W}$. Then, from

$$\sum_{f=1}^{n}\sum_{t=1}^{n}\sum_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)_{\psi}^{\psi}u_{ft}^{(1)}w_{tj}=\sum_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}_{.j}^{(1)}\right)\right)_{\psi}^{\psi},$$

it follows (3.5).

(2) Due to (2.2), it is concluded

$$a_{ij}^{\dagger,\textcircled{\dagger},W_l}=\sum_{f=1}^{n}\sum_{t=1}^{n}q_{if}^{A}w_{ft}v_{tj}^{\textcircled{\dagger},l}.$$

Applying the $\mathfrak{D}$-representations of $\mathbf{Q}_A=\mathbf{A}^\dagger\mathbf{A}=(q_{if}^{A})$ as in (3.1) and the $\mathfrak{D}$-representation of $\mathbf{V}_{\textcircled{\dagger}}=(v_{ij}^{\textcircled{\dagger},l})$ expressed by (3.4), we obtain

$$a_{ij}^{\dagger,\textcircled{\dagger},W_l}=\sum_{f=1}^{n}\sum_{t=1}^{m}\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)_{\psi}^{\psi}w_{ft}}{\sum\limits_{\psi\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_{\psi}^{\psi}}\frac{\sum\limits_{\psi\in J_{s_2,m}\{t\}}\operatorname{cdet}_t\left(\left(\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right)_{.t}\left(\check{\mathbf{v}}_{.j}\right)\right)_{\psi}^{\psi}}{\sum\limits_{\psi\in J_{s_2,m}}\left|\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right|_{\psi}^{\psi}},$$

whereat $\check{\mathbf{v}}_{.j}$ stands for jth column from $\check{\mathbf{V}}=(\mathbf{V}^{k+1})^*\mathbf{V}^k$.

Observe the (tj)th element of the matrix $\mathbf{V}_1=\left(v_{tj}^{(1)}\right)$

$$v_{tj}^{(1)}=\sum_{\psi\in J_{s_2,m}\{t\}}\operatorname{cdet}_t\left(\left(\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right)_{.t}\left(\check{\mathbf{v}}_{.j}\right)\right)_{\psi}^{\psi}$$

and the matrix $\mathbf{A}_2 = \mathbf{A}^*\mathbf{AWV}_1$. Then, (3.7) is derived using

$$\sum_{f=1}^{n}\sum_{t=1}^{m}\sum_{\psi\in J_{s,n}\{i\}} \text{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)_{\psi}^{\psi} w_{tf}v_{fj}^{(1)} = \sum_{\psi\in J_{s,n}\{i\}} \text{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}_{.j}^{(2)}\right)\right)_{\psi}^{\psi},$$

such that $\mathbf{a}_{.j}^{(2)}$ denotes jth column of $\mathbf{A}_2$. □

Also, we consider the $\mathfrak{D}$-representations of the quaternion R-**W**-CEPMP and L-**W**-CEPMP inverses.

Theorem 3.2. *Suppose that* $\mathbf{A} = (a_{ij}) \in \mathbb{H}_s^{m\times n}$ *and* $\mathbf{W} = (w_{ij}) \neq 0 \in \mathbb{H}^{n\times m}$ *satisfy* $\{\mathbf{A},\mathbf{W}\} \in \mathbb{H}_{m,n,k}$ *as well as* $\text{rank}(\mathbf{U}) = s_1$*, and* $\text{rank}(\mathbf{V}) = s_2$*.*

(1) *Then* $\mathbf{A}^{\textcircled{\dagger},\dagger,W_r} = \left(a_{ij}^{\textcircled{\dagger},\dagger,W_r}\right)$ *possesses the next* $\mathfrak{D}$*-representation*

$$a_{ij}^{\textcircled{\dagger},\dagger,W_r} = \frac{\sum\limits_{\mu\in I_{s,m}\{j\}} \text{rdet}_j\left((\mathbf{AA}^*)_{j.}(\tilde{\mathbf{a}}_{i.}^{(1)})\right)_{\mu}^{\mu}}{\sum\limits_{\mu\in I_{s_1,n}} \left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|_{\mu}^{\mu} \sum\limits_{\psi\in J_{s,m}} |\mathbf{AA}^*|_{\psi}^{\psi}}, \tag{3.9}$$

in which $\tilde{\mathbf{a}}_{i.}^{(1)}$ *denotes the* i*th row of* $\widetilde{\mathbf{A}}_1 = \mathbf{U}_1\mathbf{WAA}^*$ *and* $\mathbf{U}_1$ *is determined by* (3.6)*.*

(2) *Then* $\mathbf{A}^{\textcircled{\dagger},\dagger,W_l} = \left(a_{ij}^{\textcircled{\dagger},\dagger,W_l}\right)$ *is represented by the following* $\mathfrak{D}$*-representation*

$$a_{ij}^{\textcircled{\dagger},\dagger,W_l} = \frac{\sum\limits_{\mu\in I_{s,m}\{j\}} \text{rdet}_j\left((\mathbf{AA}^*)_{j.}(\tilde{\mathbf{a}}_{i.}^{(2)})\right)_{\mu}^{\mu}}{\sum\limits_{\mu\in I_{s_2,m}} \left|\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right|_{\mu}^{\mu} \sum\limits_{\psi\in J_{s,m}} |\mathbf{AA}^*|_{\psi}^{\psi}}, \tag{3.10}$$

wherein $\tilde{\mathbf{a}}_{i.}^{(2)}$ *indicates* i*th row from* $\widetilde{\mathbf{A}}_2 = \mathbf{WV}_1\mathbf{AA}^*$ *and* $\mathbf{V}_1$ *is determined by* (3.8)*.*

Proof. (1) Thanks to (2.3), it can be derived

$$a_{ij}^{\textcircled{\dagger},\dagger,W_r} = \sum_{t=1}^{m}\sum_{f=1}^{n} u_{if}^{\textcircled{\dagger},r} w_{ft} p_{tj}^{A}. \tag{3.11}$$

Applying the $\mathfrak{D}$-representation of $\mathbf{P}_A = \mathbf{AA}^\dagger = (p_{ij}^A)$ given by (3.2) and the $\mathfrak{D}$-representation of $\mathbf{U}^{\textcircled{\dagger}} = (u_{if}^{\textcircled{\dagger},r})$ as in (3.3), one obtains

$$a_{ij}^{\textcircled{\dagger},\dagger,W_r} = \sum_{t=1}^{m}\sum_{f=1}^{n} \frac{\sum\limits_{\mu\in I_{s_1,n}\{f\}} \text{rdet}_f\left(\left(\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right)_{f.}(\hat{\mathbf{u}}_{i.})\right)_{\mu}^{\mu}}{\sum\limits_{\mu\in I_{s_1,n}} \left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|_{\mu}^{\mu}} w_{ft} \frac{\sum\limits_{\mu\in I_{s,m}\{j\}} \text{rdet}_j((\mathbf{AA}^*)_{j.}(\ddot{\mathbf{a}}_{t.}))_{\mu}^{\mu}}{\sum\limits_{\mu\in I_{s,m}} |\mathbf{AA}^*|_{\mu}^{\mu}},$$

such that $\hat{\mathbf{u}}_{i.}$ indicates ith row from $\hat{\mathbf{U}} = \mathbf{U}^k(\mathbf{U}^{k+1})^*$ and $\ddot{\mathbf{a}}_{t.}$ is tth row from $\mathbf{AA}^* \in \mathbb{H}^{m\times m}$.

In a similar way as in (3.6), let

$$u_{if}^{(1)} := \sum_{\mu \in I_{s_1,n}\{f\}} \operatorname{rdet}_f \left(\left(\mathbf{U}^{k+1} \left(\mathbf{U}^{k+1} \right)^* \right)_{f.} (\hat{\mathbf{u}}_{i\,.}) \right)_{\mu}^{\mu}$$

stand for the (if)th element of the matrix $\mathbf{U}_1 = \left(u_{if}^{(1)} \right)$ and construct the matrix $\widetilde{\mathbf{A}}_1 = \mathbf{U}_1 \mathbf{W} \mathbf{A} \mathbf{A}^*$. Then, from

$$\sum_{t=1}^{m} \sum_{f=1}^{n} u_{if}^{(1)} w_{ft} \sum_{\mu \in I_{s,m}\{j\}} \operatorname{rdet}_j ((\mathbf{A}\mathbf{A}^*)_{j.} (\ddot{\mathbf{a}}_{t.}))_{\mu}^{\mu} = \sum_{\mu \in I_{s,m}\{j\}} \operatorname{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} (\tilde{\mathbf{a}}_{i.}^{(1)}) \right)_{\mu}^{\mu},$$

where $\tilde{\mathbf{a}}_{i.}^{(1)}$ is the ith row of $\widetilde{\mathbf{A}}_1$, it follows (3.9).

(2) Due to (2.4), one obtains

$$a_{ij}^{\dagger,\textcircled{\dagger},W_l} = \sum_{f=1}^{m} \sum_{t=1}^{m} w_{if} v_{ft}^{\textcircled{\dagger},l} p_{tj}^{A}.$$

Using the $\mathfrak{D}$-representation of $\mathbf{P}_A = \mathbf{A}\mathbf{A}^\dagger = (p_{ij}^A)$ as in (3.2) and the $\mathfrak{D}$-representation of $\mathbf{V}_{\textcircled{\dagger}} = (v_{ij}^{\textcircled{\dagger},l})$ as in (3.4), we obtain

$$a_{ij}^{\dagger,\textcircled{\dagger},W_l} = \sum_{f=1}^{m} \sum_{t=1}^{m} w_{if} \frac{\sum\limits_{\psi \in J_{s_2,m}\{f\}} \operatorname{cdet}_f \left(\left(\left(\mathbf{V}^{k+1} \right)^* \mathbf{V}^{k+1} \right)_{.f} (\check{\mathbf{v}}_{.t}) \right)_{\psi}^{\psi}}{\sum\limits_{\psi \in J_{s_2,m}} \left| \left(\mathbf{V}^{k+1} \right)^* \mathbf{V}^{k+1} \right|_{\psi}^{\psi}} \frac{\sum\limits_{\mu \in I_{s,m}\{j\}} \operatorname{rdet}_j ((\mathbf{A}\mathbf{A}^*)_{j.} (\ddot{\mathbf{a}}_{t.}))_{\mu}^{\mu}}{\sum\limits_{\mu \in I_{s,m}} |\mathbf{A}\mathbf{A}^*|_{\mu}^{\mu}},$$

wherein $\check{\mathbf{v}}_{.j}$ designates jth column from $\check{\mathbf{V}} = (\mathbf{V}^{k+1})^* \mathbf{V}^k$.

Similar as in (3.8), denote by

$$v_{ft}^{(1)} = \sum_{\psi \in J_{s_2,m}\{f\}} \operatorname{cdet}_f \left(\left(\left(\mathbf{V}^{k+1} \right)^* \mathbf{V}^{k+1} \right)_{.f} (\check{\mathbf{v}}_{.t}) \right)_{\psi}^{\psi}$$

the (ft)th element of $\mathbf{V}_1 = \left(v_{ft}^{(1)} \right)$ and determine the matrix $\widetilde{\mathbf{A}}_2 = \mathbf{W}\mathbf{V}_1\mathbf{A}\mathbf{A}^*$. Then, from

$$\sum_{f=1}^{m} \sum_{t=1}^{m} w_{if} v_{ft}^{(1)} \sum_{\mu \in I_{s,m}\{j\}} \operatorname{rdet}_j ((\mathbf{A}\mathbf{A}^*)_{j.} (\ddot{\mathbf{a}}_{t.}))_{\mu}^{\mu} = \sum_{\mu \in I_{s,m}\{j\}} \operatorname{rdet}_j \left((\mathbf{A}\mathbf{A}^*)_{j.} (\tilde{\mathbf{a}}_{i.}^{(2)}) \right)_{\mu}^{\mu},$$

wherein $\tilde{\mathbf{a}}_{i.}^{(2)}$ represents the ith row of $\widetilde{\mathbf{A}}_2$, it follows (3.10). □

Finally, we give the $\mathfrak{D}$-representations of the quaternion L-**W**-MPCEPMP and R-**W**-MPCEPMP inverses.

Theorem 3.3. *Suppose that* $\mathbf{A}=(a_{ij})\in\mathbb{H}_s^{m\times n}$ *and* $\mathbf{W}=(w_{ij})\neq 0\in\mathbb{H}^{n\times m}$ *satisfy* $\{\mathbf{A},\mathbf{W}\}\in\mathbb{H}_{m,n,k}$ *as well as* $\operatorname{rank}(\mathbf{U})=s_1$, *and* $\operatorname{rank}(\mathbf{V})=s_2$.

(1) *Then* $\mathbf{A}^{\dagger,\oplus,\dagger,W_r}=\left(a_{ij}^{\dagger,\oplus,W_r}\right)$ *has the following* $\mathfrak{D}$*-representations*

$$a_{ij}^{\dagger,\oplus,\dagger,W_r}=\frac{\sum\limits_{\mu\in I_{s,m}\{j\}}\operatorname{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}(\boldsymbol{\phi}_{i.}^{(1)})\right)_\mu^\mu}{\sum\limits_{\mu\in I_{s_1,n}}\left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|_\mu^\mu\left(\sum\limits_{\psi\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\psi^\psi\right)^2} \tag{3.12}$$

$$=\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\boldsymbol{\psi}_{.j}^{(1)}\right)\right)_\psi^\psi}{\sum\limits_{\mu\in I_{s_1,n}}\left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|_\mu^\mu\left(\sum\limits_{\psi\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\psi^\psi\right)^2}, \tag{3.13}$$

where $\boldsymbol{\phi}_{i.}^{(1)}$ *means* i*th row of* $\boldsymbol{\Phi}_1=\mathbf{Q}_1\mathbf{U}_1\mathbf{W}\mathbf{A}\mathbf{A}^*$ *and* $\boldsymbol{\psi}_{.j}^{(1)}$ *is the* j*th column of* $\boldsymbol{\Psi}_1=\mathbf{A}^*\mathbf{A}\mathbf{U}_1\mathbf{W}\mathbf{P}_1$, *and* $\mathbf{U}_1$ *is defined by* (3.6). *Here* $\mathbf{Q}_1=(q_{if}^{(1)})$ *and* $\mathbf{P}_1=(p_{lj}^{(1)})$ *are such that*

$$q_{if}^{(1)}=\sum_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)_\psi^\psi, \tag{3.14}$$

$$p_{lj}^{(1)}=\sum_{\mu\in I_{s,m}\{j\}}\operatorname{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{l.}))_\mu^\mu, \tag{3.15}$$

$\dot{\mathbf{a}}_{.f}$ *means* f*th column from* $\mathbf{A}^*\mathbf{A}$, *and* $\ddot{\mathbf{a}}_{l.}$ *means* l*th row lying in* $\mathbf{A}\mathbf{A}^*\in\mathbb{H}^{m\times m}$.

(2) *Then the following* $\mathfrak{D}$*-representations of* $\mathbf{A}^{\dagger,\oplus,\dagger,W_l}=\left(a_{ij}^{\dagger,\oplus,W_l}\right)$ *hold:*

$$a_{ij}^{\dagger,\oplus,\dagger,W_l}=\frac{\sum\limits_{\mu\in I_{s,m}\{j\}}\operatorname{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}(\boldsymbol{\phi}_{i.}^{(2)})\right)_\mu^\mu}{\sum\limits_{\psi\in J_{s_2,m}}\left|\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right|_\psi^\psi\left(\sum\limits_{\psi\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\psi^\psi\right)^2} \tag{3.16}$$

$$=\frac{\sum\limits_{\psi\in J_{s,n}\{i\}}\operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\boldsymbol{\psi}_{.j}^{(2)}\right)\right)_\psi^\psi}{\sum\limits_{\psi\in J_{s_2,m}}\left|\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right|_\psi^\psi\left(\sum\limits_{\psi\in J_{s,n}}|\mathbf{A}^*\mathbf{A}|_\psi^\psi\right)^2}, \tag{3.17}$$

where $\boldsymbol{\phi}_{i.}^{(2)}$ *signifies* i*th row of* $\boldsymbol{\Phi}_2=\mathbf{Q}_1\mathbf{W}\mathbf{V}_1\mathbf{A}\mathbf{A}^*$ *and* $\boldsymbol{\psi}_{.j}^{(2)}$ *stands for* j*th column of* $\boldsymbol{\Psi}_2=\mathbf{A}^*\mathbf{A}\mathbf{W}\mathbf{V}_1\mathbf{P}_1$; $\mathbf{V}_1$, $\mathbf{Q}_1=(q_{if}^{(1)})$, *and* $\mathbf{P}_1=(p_{lj}^{(1)})$ *are determined by* (3.8), (3.14), (3.15), *respectively.*

Proof. (1) Thanks to (2.5), it follows that

$$a_{ij}^{\dagger,\oplus,\dagger,W_r}=\sum_{f=1}^n\sum_{t=1}^n\sum_{l=1}^m q_{if}^A u_{ft}^{\oplus,r}w_{tl}p_{lj}^A.$$

According to the $\mathfrak{D}$-representation of $\mathbf{Q}_A = \mathbf{A}^\dagger\mathbf{A} = (q^A_{if})$ expressed by (3.1), of $\mathbf{P}_A = \mathbf{A}\mathbf{A}^\dagger = (p^A_{ij})$ as in (3.2) and of $\mathbf{U}^{\oplus} = (u^{\oplus,r}_{ft})$ given by (3.3), we obtain

$$a^{\oplus,\dagger,W_r}_{ij} = \sum_{f=1}^{n}\sum_{t=1}^{n}\sum_{l=1}^{m} \frac{\sum\limits_{\psi\in J_{s,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)^{\psi}_{\psi}}{\sum\limits_{\psi\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|^{\psi}_{\psi}} \frac{\sum\limits_{\mu\in I_{s_1,n}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right)_{t.}\left(\hat{\mathbf{u}}_{f\,.}\right)\right)^{\mu}_{\mu}}{\sum\limits_{\mu\in I_{s_1,n}} \left|\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right|^{\mu}_{\mu}} w_{tl}$$
$$\times \frac{\sum\limits_{\mu\in I_{s,m}\{j\}} \operatorname{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{l.}))^{\mu}_{\mu}}{\sum\limits_{\mu\in I_{s,m}} |\mathbf{A}\mathbf{A}^*|^{\mu}_{\mu}},$$

where $\hat{\mathbf{u}}_{f\,.}$ symbolizes fth row of $\hat{\mathbf{U}} = \mathbf{U}^k(\mathbf{U}^{k+1})^*$, $\dot{\mathbf{a}}_{.f}$ is fth column of $\mathbf{A}^*\mathbf{A}$, and $\ddot{\mathbf{a}}_{l.}$ is lth row of $\mathbf{A}\mathbf{A}^* \in \mathbb{H}^{m\times m}$. It is evident that $\sum\limits_{\psi\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|^{\psi}_{\psi} = \sum\limits_{\mu\in I_{s,m}} |\mathbf{A}\mathbf{A}^*|^{\mu}_{\mu}$.

Again, let

$$u^{(1)}_{ft} := \sum_{\mu\in I_{s_1,n}\{t\}} \operatorname{rdet}_t\left(\left(\mathbf{U}^{k+1}\left(\mathbf{U}^{k+1}\right)^*\right)_{t.}\left(\hat{\mathbf{u}}_{f\,.}\right)\right)^{\mu}_{\mu}$$

be the (ft)th element of the matrix $\mathbf{U}_1 = \left(u^{(1)}_{ft}\right)$, and denote by

$$q^{(1)}_{if} = \sum_{\psi\in J_{s,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)^{\psi}_{\psi},$$
$$p^{(1)}_{lj} = \sum_{\mu\in I_{s,m}\{j\}} \operatorname{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{l.}))^{\mu}_{\mu},$$

the (if)th and (lj)th elements of the matrices $\mathbf{Q}_1 = (q^{(1)}_{if})$ and $\mathbf{P}_1 = (p^{(1)}_{lj})$, respectively.

If we construct the matrix $\mathbf{\Phi}_1 = \mathbf{Q}_1\mathbf{U}_1\mathbf{W}\mathbf{A}\mathbf{A}^*$, then, from

$$\sum_{f=1}^{n}\sum_{t=1}^{n}\sum_{l=1}^{m} q^{(1)}_{if} u^{(1)}_{ft} w_{tl} \sum_{\mu\in I_{s,m}\{j\}} \operatorname{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{l.}))^{\mu}_{\mu} = \sum_{\mu\in I_{s,m}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}(\boldsymbol{\phi}^{(1)}_{i.})\right)^{\mu}_{\mu},$$

where $\boldsymbol{\phi}^{(1)}_{i.}$ stands for ith row of $\mathbf{\Phi}_1$, it follows (3.12).

If we determine the matrix $\mathbf{\Psi}_1 = \mathbf{A}^*\mathbf{A}\mathbf{U}_1\mathbf{W}\mathbf{P}_1$, then, from

$$\sum_{f=1}^{n}\sum_{t=1}^{n}\sum_{l=1}^{m} \sum_{\psi\in J_{s,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\dot{\mathbf{a}}_{.f}\right)\right)^{\psi}_{\psi} u^{(1)}_{ft} w_{tl} p^{(1)}_{lj} = \sum_{\psi\in J_{s,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\boldsymbol{\psi}^{(1)}_{.j}\right)\right)^{\psi}_{\psi},$$

wherein $\boldsymbol{\psi}^{(1)}_{.j}$ denotes jth column from $\mathbf{\Psi}_1$, it follows (3.13).

(2) Due to (2.5), it follows

$$a^{\dagger,\oplus,\dagger,W_l}_{ij} = \sum_{f=1}^{n}\sum_{t=1}^{m}\sum_{g=1}^{m} q^A_{if} w_{ft} v^{\oplus,l}_{tg} p^A_{gj}.$$

Using $\mathfrak{D}$-representations of $\mathbf{Q}_A = \mathbf{A}^\dagger\mathbf{A} = (q_{if}^A)$ given by (3.1), of $\mathbf{V}_{\oplus} = (v_{tg}^{\oplus,l})$ as in (3.4), and of $\mathbf{P}_A = \mathbf{A}\mathbf{A}^\dagger = (p_{gj}^A)$ as in (3.2), we obtain the representation

$$a_{ij}^{\dagger,\oplus,W_l} = \sum_{f=1}^{n}\sum_{t=1}^{m}\sum_{g=1}^{m} \frac{\sum\limits_{\psi\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.f})\right)_\psi^\psi}{\sum\limits_{\psi\in J_{s,n}} |\mathbf{A}^*\mathbf{A}|_\psi^\psi} w_{ft} \frac{\sum\limits_{\psi\in J_{s_2,m}\{t\}} \mathrm{cdet}_t\left(\left(\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right)_{.t}(\check{\mathbf{v}}_{.g})\right)_\psi^\psi}{\sum\limits_{\psi\in J_{s_2,m}} \left|\left(\mathbf{V}^{k+1}\right)^*\mathbf{V}^{k+1}\right|_\psi^\psi}$$
$$\times \frac{\sum\limits_{\mu\in I_{s,m}\{j\}} \mathrm{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{g.}))_\mu^\mu}{\sum\limits_{\mu\in I_{s,m}} |\mathbf{A}\mathbf{A}^*|_\mu^\mu},$$

in which $\check{\mathbf{v}}_{.g}$ is gth column of $\check{\mathbf{V}} = (\mathbf{V}^{k+1})^*\mathbf{V}^k$.

Determine $\mathbf{V}_1 = \left(v_{ft}^{(1)}\right)$ by (3.8), and the matrices $\mathbf{Q}_1 = (q_{if}^{(1)})$ and $\mathbf{P}_1 = (p_{gj}^{(1)})$ by (3.14) and (3.15), respectively.

If we construct the matrix $\mathbf{\Phi}_2 = \mathbf{Q}_1\mathbf{W}\mathbf{V}_1\mathbf{A}\mathbf{A}^*$, then, from

$$\sum_{f=1}^{n}\sum_{t=1}^{m}\sum_{g=1}^{m} q_{if}^{(1)} w_{ft} v_{tg}^{(1)} \sum_{\mu\in I_{s,m}\{j\}} \mathrm{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{g.}))_\mu^\mu = \sum_{\mu\in I_{s,m}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}(\boldsymbol{\phi}_{i.}^{(2)})\right)_\mu^\mu,$$

where $\boldsymbol{\phi}_{i.}^{(2)}$ represents ith row involved in $\mathbf{\Phi}_2$, one obtains (3.16).

If we determine the matrix $\mathbf{\Psi}_2 = \mathbf{A}^*\mathbf{A}\mathbf{W}\mathbf{V}_1\mathbf{P}_1$, then (3.17) follows from

$$\sum_{f=1}^{n}\sum_{t=1}^{m}\sum_{g=1}^{m} \sum_{\psi\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.f})\right)_\psi^\psi w_{ft} v_{fg}^{(1)} p_{gj}^{(1)} = \sum_{\psi\in J_{s,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\boldsymbol{\psi}_{.j}^{(2)}\right)\right)_\psi^\psi,$$

wherein $\boldsymbol{\psi}_{.j}^{(2)}$ stands for jth column of $\mathbf{\Psi}_2$. □

4 Illustrative example

Consider

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{i} & 0 \\ \mathbf{k} & 1 & \mathbf{i} \\ 1 & 0 & 0 \\ 1 & -\mathbf{k} & -\mathbf{j} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{k} & 0 & \mathbf{i} & 0 \\ -\mathbf{j} & \mathbf{k} & 0 & 1 \\ 0 & 1 & 0 & -\mathbf{k} \end{bmatrix}.$$

Since

$$\mathbf{V} = \mathbf{A}\mathbf{W} = \begin{bmatrix} -\mathbf{k} & -\mathbf{j} & 0 & \mathbf{i} \\ -1-\mathbf{j} & \mathbf{i}+\mathbf{k} & \mathbf{j} & 1+\mathbf{j} \\ \mathbf{k} & 0 & \mathbf{i} & 0 \\ -\mathbf{i}+\mathbf{k} & 1-\mathbf{j} & \mathbf{i} & \mathbf{i}-\mathbf{k} \end{bmatrix}, \quad \mathbf{U} = \mathbf{W}\mathbf{A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & 0 \\ 0 & \mathbf{k} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\mathrm{rank}\,(\mathbf{A}\mathbf{W}) = 3$, $\mathrm{rank}\,(\mathbf{A}\mathbf{W})^3 = \mathrm{rank}\,(\mathbf{A}\mathbf{W})^2 = 2$, $\mathrm{rank}\,(\mathbf{W}\mathbf{A})^2 = \mathrm{rank}\,(\mathbf{W}\mathbf{A}) = 2$, it is computed $k_1 = \max\{\mathrm{Ind}(\mathbf{A}\mathbf{W}), \mathrm{Ind}(\mathbf{W}\mathbf{A})\} = 2$.

The right $\mathbf{W}$-MPCEP inverse can be found by calculations inducted by Eq. (3.5).

1. Compute the matrices

$$\widehat{\mathbf{U}} = \mathbf{U}^2(\mathbf{U}^3)^* = \begin{bmatrix} -4\mathbf{i}+\mathbf{k} & -1-\mathbf{j} & 0 \\ 1-2\mathbf{j} & -\mathbf{k} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{U}^3\left(\mathbf{U}^3\right)^* = \begin{bmatrix} 6 & -2\mathbf{i}-\mathbf{k} & 0 \\ 2\mathbf{i}+\mathbf{k} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the value

$$\sum_{\mu \in I_{2,3}} \left|\mathbf{U}^3\left(\mathbf{U}^3\right)^*\right|_{\mu}^{\mu} = \det \begin{bmatrix} 6 & -2\mathbf{i}-\mathbf{k} \\ 2\mathbf{i}+\mathbf{k} & 1 \end{bmatrix} = 1.$$

2. Find the matrix $\mathbf{U}_1$ by (3.6). So,

$$\mathbf{U}_1 = \begin{bmatrix} -\mathbf{i} & 1 & 0 \\ 0 & -\mathbf{k} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3. Compute the matrix

$$\mathbf{A}_1 = \mathbf{A}^*\mathbf{A}\mathbf{U}_1\mathbf{W} = \begin{bmatrix} 2\mathbf{j} & \mathbf{k} & 3 & 1 \\ -3\mathbf{i} & 1 & 2\mathbf{k} & -\mathbf{k} \\ -2 & 0 & 2\mathbf{j} & 0 \end{bmatrix}, \quad \mathbf{A}^*\mathbf{A} = \begin{bmatrix} 3 & -2\mathbf{k} & -2\mathbf{j} \\ 2\mathbf{k} & 3 & 2\mathbf{i} \\ 2\mathbf{j} & -2\mathbf{i} & 2 \end{bmatrix}.$$

Since $\text{rank}(\mathbf{A}) = 3$, then

$$\sum_{\psi \in J_{3,3}} |\mathbf{A}^*\mathbf{A}|_{\psi}^{\psi} = \det \begin{bmatrix} 3 & -2\mathbf{k} & -2\mathbf{j} \\ 2\mathbf{k} & 3 & 2\mathbf{i} \\ 2\mathbf{j} & -2\mathbf{i} & 2 \end{bmatrix} = 2.$$

4. Finally, find the matrix $\mathbf{A}^{\dagger,\textcircled{\dagger},W_r}$ by (3.5). So,

$$\mathbf{A}^{\dagger,\textcircled{\dagger},W_r} = \begin{bmatrix} 0 & \mathbf{k} & 1 & 1 \\ -\mathbf{i} & 1 & 0 & -\mathbf{k} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, we can find:

- by Eq. (3.7),

$$\mathbf{A}^{\dagger,\textcircled{\dagger},W_l} = \begin{bmatrix} 0 & \mathbf{k} & 1 & 1 \\ -\mathbf{i} & 1 & 0 & -\mathbf{k} \\ 0.8\mathbf{j} & 0.4\mathbf{j}-0.8\mathbf{k} & 0.4\mathbf{j} & -0.8+0.4\mathbf{j} \end{bmatrix};$$

- by Eqs. (3.9) and (3.10), respectively,

$$\mathbf{A}^{\textcircled{\dagger},\dagger,W_r} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\mathbf{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}^{\textcircled{\dagger},\dagger,W_l} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\mathbf{i} & 0 & 0 & 0 \\ 0.8\mathbf{j} & 0 & 0.2\mathbf{j} & 0 \end{bmatrix}.$$

According to the given matrices, we have $\mathbf{Q}_1 = 2\mathbf{I}$ by Eq. (3.14). From this it follows that $\mathbf{A}^{\dagger,\oplus,\dagger,W_r} = \mathbf{A}^{\oplus,\dagger,W_r}$ and $\mathbf{A}^{\dagger,\oplus,\dagger,W_l} = \mathbf{A}^{\oplus,\dagger,W_l}$.

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