Outer inverses in semigroups belonging to the prescribed Green's equivalence classes

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Abstract. We provide existence criteria and characterizations for outer inverses in a semigroup belonging to the prescribed Green's $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes. These results generalize the well-known problem of finding outer inverses of a matrix over a field with the prescribed range or/and null space. We show that Mary's inverse along an element, Drazin's (b,c)-inverse, and Bott-Duffin (e,f)-inverse of a given element are just three different ways of representing the same notion – the outer inverse of this element belonging to the prescribed Green's $\mathscr{H}$-class. Hence, outer inverses belonging to the prescribed Green's $\mathscr{R}$- and $\mathscr{L}$-classes represent extensions of (b,c)-inverses and inverses along an element. We provide an overview of other such extensions that have emerged recently and compare them with the extensions introduced in this paper.

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1. Introduction

The study of generalized inverses has a very long and rich history. Penrose's approach to generalized inverses made it possible to extend the concept of the Moore-Penrose inverse (as well as certain related concepts) from matrices and operators to more general algebraic structures. As a result, various combinations of Moore-Penrose equations and related inverses were studied in the settings of (involutive) semigroups, rings, and other related structures. In addition to Moore-Penrose inverses, many other generalized inverses have been extensively studied, such as least-squares and minimum-norm inverses, the Drazin inverse, the group inverse, core and dual core inverses, and

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others. At the root of all these generalized inverses lie inner and outer inverses. As known, inner and outer inverses play an essential role in matrix theory. In particular, inner inverses play a key role in solving linear matrix equations and systems of linear equations. In addition, outer inverses are widely used in iterative methods for solving nonlinear equations, stable approximations of ill-posed problems, solving linear and nonlinear problems involving rank-deficient generalized inverses, statistics, and other areas.

Two new and important types of generalized inverses have emerged recently: the inverse along an element, introduced by X. Mary [44], and the (b,c)-inverse, introduced by M. Drazin [24]. Both are outer inverses which, as their special cases, include the Moore-Penrose inverse, group inverse, Drazin inverse, core and dual core inverses, and other important types of generalized inverses. In addition, (b,c)-inverses originated as semigroup-theoretical counterparts of outer inverses of matrices with the prescribed range and null space (cf. [3, Ch. 2, Sec. 6; Ch. 7, Sec. 4]).

After their introduction, an impressive number of papers dealt with (b,c)-inverses and inverses along an element. They have been most studied in the contexts of rings [4,5,16,36,38–40,53,67,68,75,80–84] and semigroups [2,14,16,24–27,44–46,49], and they have been also studied in the context of Banach algebras [8,9,45], residuated semigroups and quantales [33], matrices over a field [6,15,61,70], matrices over a ring [37], tensors [59], and fuzzy matrices with entries in a complete residuated lattice [18]. Core and dual core inverses were studied in [1,43,57,69,76], in the contexts of matrices and rings, and inner and outer inverses with prescribed idempotents and ideals have been considered in [13,22,23,34,35,52,54], in the contexts of rings, Banach algebras and operator algebras. For further information on the results concerning outer inverses of matrices with the prescribed range and null space we refer to [3,58,61,66,71,73,77] and other sources cited there.

Mary [44] defined and studied inverses along an element using one of the most powerful tools of the semigroup theory – Green's relations. They have also been used in his other studies [45–49]. However, to solve equations uniquely, Mary mainly emphasized relation $\mathscr{H}$. Therefore, only a part of what the theory of Green's relations offers has been used in these papers. Our goal is to show the full strength of that theory here, notably by using the one-sided relations $\mathscr{R}$ and $\mathscr{L}$. The general problem we discuss is the existence criteria and characterization of outer and inner inverses in semigroups belonging to the prescribed Green's $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes. This problem generalizes the problem of determining the existence criteria and characterization of the outer inverse of matrices over a field with the prescribed range and/or null space. In addition, we discuss the existence criteria and characterization of inner inverses belonging to the prescribed principal right, left, and quasi-ideals. We recover that Drazin's (b,c)-inverse and Mary's inverse along an element are essentially the same (see also [47, Proposition 1.4, Corollary 1.1], [48], [75]). Both are outer inverses that belong to a prescribed Green's $\mathscr{H}$-class, and the only difference is how that class is represented.

One of the ways we use here for setting criteria of existence of outer and inner inverses with prescribed properties is through solvability of certain equations in a semigroup, and the solutions of these equations are used to characterize these inverses. In most cases, the criteria are set using only one equation, and even when using a system of two equations, or in one case three, we show that they have the same sets of solutions, so it is enough to solve only one of them. This fact can be extremely important when generalized inverses of matrices are computed using neural networks. Such an approach emerged in the 1990s and has been gaining momentum in recent years. As pointed out in [74], the complexity of the neural network increases as the number of matrix equations increases, and decreasing the number of matrix equations needed to compute the desired generalized inverse will reduce this complexity. As a result, the architecture and implementation of the neural network will be simplified.

The paper is organized as follows. In Section 2, we introduce basic concepts and notation concerning semigroups and generalized inverses and outline the main results that will be used in the further text, as well as examples that illustrate Green's relations in some important semigroups, including semigroups of matrices over a field and a semiring. Then, in Section 3, we provide the existence conditions and characterizations for outer inverses of a given element belonging to the prescribed Green's $\mathscr{R}$- and $\mathscr{L}$-classes, as well as for inner inverses belonging to the prescribed right and left principal ideals and reflexive generalized inverses belonging to the prescribed Green's $\mathscr{R}$- and $\mathscr{L}$-classes. In Section 4, we show that Mary's inverse along an element, Drazin's (b,c)-inverse, and Bott-Duffin (e,f)-inverse of a given element are just three different ways in representing the same thing – the outer inverse of this element belonging to the corresponding $\mathscr{H}$-class, whereas the Djordjević and Wei's outer inverse with prescribed idempotents is their proper special case. We provide some new existence criteria and characterizations for (b,c)-inverses and inverses along an element, as well as for group inverses, and give new proofs of some known results concerning these inverses. In Section 5, we give an overview of recently emerged extensions of (b,c)-inverses and inverses along an element and compare them with the concepts introduced in this paper. In Section 6 we present computational consequences of the results obtained in the previous sections. In particular, we provide a table that links various types of equations with corresponding generalized inverses. Finally, in Section 7 we give concluding remarks and visions of our future research.

2. Preliminaries: Green's equivalences, generalized inverses

This section introduces basic concepts and notation regarding semigroups and generalized inverses. For undefined notions and notation regarding semigroups we refer to [19,32], and for those regarding generalized inverses in different contexts we refer to [3,7,12,21,66].

It is worth noting that many of the statements proved here have their dual statements (for the concept of duality, we refer to [19, page 5]). For the sake of completeness, these dual statements will be stated, but their proofs will be omitted since any proof of a statement that has a dual can be transformed into a proof of its dual statement.

The results obtained in this paper are very general and can be applied to many specific mathematical systems such as partial and full mappings, linear operators, matrices over a field, ring or semiring, tensors, etc.

Let S is a semigroup. By $S^{\mathbb{1}}$ we denote the semigroup $S\cup\{\mathbb{1}\}$ arising from S by the adjunction of an identity element $\mathbb{1}$, unless S already has an identity, in which case $S^{\mathbb{1}}=S$. The smallest left ideal of S containing an element $a\in S$ is $S^{\mathbb{1}}a=Sa\cup\{a\}$, which is conveniently denoted by $L(a)$ and called the *principal left ideal of S generated by a*. Analogously, the smallest right ideal of S containing a is $aS^{\mathbb{1}}=aS\cup\{a\}$, and it is denoted by $R(a)$ and called the *principal right ideal of S generated by a*, and the smallest ideal of S containing a is $S^{\mathbb{1}}aS^{\mathbb{1}}=aS\cup\{a\}\cup\{a^2\}$, and it is denoted by $J(a)$ and called the *principal ideal of S generated by a*. Recall that a *quasi-ideal* of S is a subset Q of S satisfying $SQ\cap QS\subseteq Q$. Equivalently, Q is a quasi-ideal if and only if it is an intersection of a left and a right ideal of S. For any $a\in S$, the smallest quasi-ideal of S containing a, which is denoted by $Q(a)$ and called the *principal quasi-ideal of S generated by a*, is represented by $Q(a)=L(a)\cap R(a)$.

An equivalence $\mathscr{L}$ on S is defined by the rule: $a\,\mathscr{L}\,b$ if and only if a and b generate the same principal left ideal, i.e., if and only if $L(a)=L(b)$. Similarly, equivalences $\mathscr{R}$ and $\mathscr{J}$ on S is defined by the rules: $a\,\mathscr{R}\,b$ if and only if a and b generate the same principal right ideal, i.e., if and only if $R(a)=R(b)$, and $a\,\mathscr{J}\,b$ if and only if a and b generate the same principal ideal, i.e., if and only if $J(a)=J(b)$. In other words,

$$a\,\mathscr{L}\,b \;\Leftrightarrow\; (\exists u,v\in S^{\mathbb{1}})\, a=vb,\ b=ua, \tag{1}$$

$$a\,\mathscr{R}\,b \;\Leftrightarrow\; (\exists s,t\in S^{\mathbb{1}})\ \ a=bt,\ b=as, \tag{2}$$

$$a\,\mathscr{J}\,b \;\Leftrightarrow\; (\exists u,v,s,t\in S^{\mathbb{1}})\ \ a=ubv,\ b=sat. \tag{3}$$

It is not hard to verify that $\mathscr{L}$ is a right congruence and $\mathscr{R}$ is a left congruence. The intersection of $\mathscr{L}$ and $\mathscr{R}$ is the equivalence denoted by $\mathscr{H}$, i.e., $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$. In addition, the relations $\mathscr{L}$ and $\mathscr{R}$ commute, that is, $\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L}$, where, as is usual, $\circ$ denotes the composition of relations, so $\mathscr{D}=\mathscr{L}\circ\mathscr{R}=\mathscr{R}\circ\mathscr{L}$ is the smallest equivalence on S containing both $\mathscr{L}$ and $\mathscr{R}$ (it is the join of $\mathscr{L}$ and $\mathscr{R}$ in the lattice of all equivalences on S). Since $\mathscr{L}\subseteq\mathscr{J}$ and $\mathscr{R}\subseteq\mathscr{J}$, it follows $\mathscr{D}\subseteq\mathscr{J}$. Equivalences $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, $\mathscr{H}$ and $\mathscr{D}$ are known as *Green's equivalences*. The $\mathscr{L}$-class (resp. $\mathscr{R}$-class, $\mathscr{H}$-class, $\mathscr{D}$-class) containing the element a will be denoted by L_a (resp. R_a, H_a, D_a).

Each $\mathscr{D}$-class of a semigroup S is the union of $\mathscr{L}$-classes contained in it and the union of $\mathscr{R}$-classes contained in it. In general, the intersection of an $\mathscr{L}$-class and an $\mathscr{R}$-class may be empty, but the situation is different if these two classes are contained in the same $\mathscr{D}$-class. Namely, for any $a, b \in S$ we have

$$a \,\mathscr{D}\, b \;\Leftrightarrow\; R_a \cap L_b \neq \emptyset \;\Leftrightarrow\; L_a \cap R_b \neq \emptyset. \tag{4}$$

This analysis makes it possible to visualize the $\mathscr{D}$-class as what Clifford and Preston [19] have called an 'eggbox', in which rows represent $\mathscr{R}$-classes, columns represent $\mathscr{L}$-classes, and cells represent $\mathscr{H}$-classes (cf. Figure 1 (a)).

The restriction of a mapping ϕ to a subset X of its domain is denoted by $\phi|_X$. Let S be a semigroup and $a \in S$. The *inner left translation* λ_a and the *inner right translation* ϱ_a determined by a are mappings of S into itself defined by the rules $\lambda_a(x) = ax$ and $\varrho_a(x) = xa$, for each $x \in S$.

We highlight the fundamental results concerning Green's equivalences.

Lemma 2.1 (Green's lemma). *Let a and b be $\mathscr{R}$-related elements of a semigroup S, and let $s, t \in S^{\mathbb{1}}$ be elements such that $as = b$ and $bt = a$. Then the right translations $\varrho_s|_{L_a}$ and $\varrho_t|_{L_b}$ are mutually inverse bijections from L_a onto L_b and L_b onto L_a, respectively.*

In addition, they preserve $\mathscr{R}$-classes, i.e., $x \,\mathscr{R}\, \varrho_s(x)$ and $y \,\mathscr{R}\, \varrho_t(y)$, for all $x \in L_a$, $y \in L_b$.

Lemma 2.2 (Green's lemma). *Let a and b be $\mathscr{L}$-related elements of a semigroup S, and let $u, v \in S^{\mathbb{1}}$ be elements such that $ua = b$ and $vb = a$. Then the left translations $\lambda_u|_{R_a}$ and $\lambda_v|_{R_b}$ are mutually inverse bijections from R_a onto R_b and R_b onto R_a, respectively.*

In addition, they preserve $\mathscr{L}$-classes, i.e., $x \,\mathscr{L}\, \lambda_u(x)$ and $y \,\mathscr{L}\, \lambda_v(y)$, for all $x \in R_a$, $y \in R_b$.

Lemmas 2.1 and 2.2 are visualised by Figures 1 (b) and (c), respectively.

Proposition 2.3. *Every idempotent e of a semigroup S is a left identity for R_e and a right identity for L_e.*

Theorem 2.4 (Green's Theorem). *If H is an $\mathscr{H}$-class of a semigroup S, then either $H^2 \cap H = \emptyset$ or H is a maximal subgroup of S.*

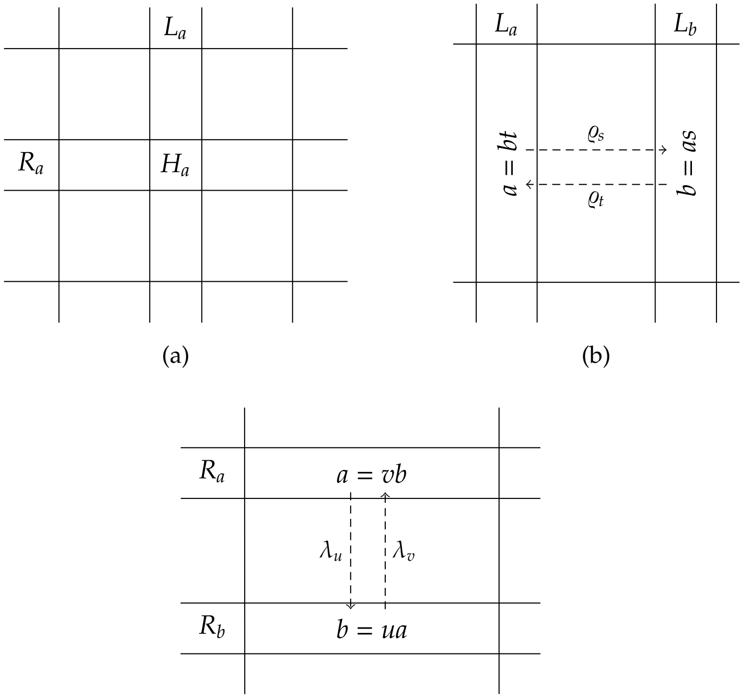
Theorem 2.5 (Miller-Clifford's theorem). *Let S be a semigroup and let $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an idempotent.*

In the case when $ab \in R_a \cap L_b$, ab is called a *trace product* (cf. [44,51,56]).

In the sequel we present characterizations of Green's relations on some important semigroups.

Example 2.6. *(Full transformation semigroup)* Let $\mathscr{T}_X$ be the *full transformation semigroup* on a set X, consisting of all maps from X into X. Subsemigroups of full transformation semigroups are called *transformation semigroups*. If X is a finite set with n elements, it will be represented as $X = \{1, 2, \ldots, n\}$, and a map $\alpha \in \mathscr{T}_X$ will be represented as $\alpha = (\alpha_1\alpha_2 \ldots \alpha_n)$, where $\alpha_i = \alpha(i)$, for each $i \in X$.

Although in the semigroup theory, it is usual for a mapping to be written on the right, in order to harmonize the notation with the one used in linear algebra, we will write mappings on the left. The same way of



(c)

FIGURE 1. Visualisations of the 'eggbox' ((a)) and Green's lemmas ((b) and (c))

writing is also used in [30]. The composition of two maps $\alpha, \beta \in \mathcal{T}_X$, written multiplicatively, is then defined by $(\alpha\beta)(x) = \alpha(\beta(x))$, for every $x \in X$.

For $\alpha \in \mathcal{T}_X$ let $\mathrm{im}(\alpha)$ denote the *image* or *range* of α and $\ker(\alpha)$ the *kernel* of α, which are defined by

$$\mathrm{im}(\alpha) = \{y \in X \,|\, (\exists x \in X)\, \alpha(x) = y\}, \ \ker(\alpha) = \{(x, y) \in X \times X \,|\, \alpha(x) = \alpha(y)\},$$

and let $\mathrm{rank}\,(\alpha)$ denote the *rank* of α, which is defined by $\mathrm{rank}\,(\alpha) = |\,\mathrm{im}(\alpha)|$, i.e., as the cardinality of $\mathrm{im}(\alpha)$. Clearly, $\mathrm{rank}(\alpha) = |X/\ker(\alpha)|$, where $X/\ker(\alpha)$ denotes the factor set of X with respect to $\ker(\alpha)$. For arbitrary $\alpha, \beta \in \mathcal{T}_X$ we have

$$\mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\alpha), \ \ \ker(\beta) \subseteq \ker(\alpha\beta), \ \ \mathrm{rank}(\alpha\beta) \leqslant \min\{\mathrm{rank}(\alpha), \mathrm{rank}(\beta)\}.$$

According to [30, Theorem 4.5.1] (see also [19, §2.2] or [32, Exercise 2.6.16]) it follows

$$\begin{aligned} \alpha\,\mathscr{R}\,\beta &\Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta), \\ \alpha\,\mathscr{L}\,\beta &\Leftrightarrow \ker(\alpha) = \ker(\beta), \\ \alpha\,\mathscr{H}\,\beta &\Leftrightarrow \operatorname{im}(\alpha) = \operatorname{im}(\beta) \ \wedge\ \ker(\alpha) = \ker(\beta), \\ \alpha\,\mathscr{D}\,\beta &\Leftrightarrow \alpha\,\mathscr{J}\,\beta \ \Leftrightarrow\ \operatorname{rank}(\alpha) = \operatorname{rank}(\beta). \end{aligned} \tag{5}$$

By writing the mapping on the right, we also change the definition of multiplication in the full transformation semigroup and obtain the dual semigroup. In this case, Green's relations $\mathscr{L}$ and $\mathscr{R}$ interchange their roles, and for that reason, our characterizations of $\mathscr{L}$ and $\mathscr{R}$ in the full transformation semigroup differ from those given in [19] and [32]. The same remark also applies to characterizations of $\mathscr{L}$ and $\mathscr{R}$ in the semigroup of linear transformations, considered in the following example.

Example 2.7. *(Semigroup of linear transformations)* Let $\mathscr{LT}(V)$ be the *semigroup of linear transformations* of a vector space V over a field, under the multiplication defined in Example 2.6. For a linear transformation $\alpha \in \mathscr{LT}(V)$ we define the *image* or *range* $\operatorname{im}(\alpha)$ as in Example 2.6, whereas the *kernel* or *null space* $\ker(\alpha)$ is defined by $\ker(\alpha) = \{\boldsymbol{v} \in V \mid \alpha(\boldsymbol{v}) = \mathbf{0}\}$, and the *rank* of α is defined by $\operatorname{rank}(\alpha) = \dim(\operatorname{im}\alpha)$, i.e., as the dimension of the subspace $\operatorname{im}(\alpha)$.

Based on [19, Exercise 2.2.6] or [32, Exercise 2.6.19], Green's equivalences on $\mathscr{LT}(V)$ can be given by the same formulas as in (5). Note that the $\mathscr{H}$-class of α is a group if and only if $\operatorname{im}(\alpha) \cap \ker(\alpha) = \{\mathbf{0}\}$.

It is essential to point out that in many sources, one can find the claim that matrices, except for the square ones, do not form a semigroup. However, this is only partially true. Matrices of arbitrary type form a partial semigroup. As we show in Example 2.8, utilizing the standard semigroup-theoretical method, it can be converted into a full semigroup without spoiling anything related to matrices.

Example 2.8. *(Semigroup of matrices over a field)* Let $M(\mathbb{F})$ be the set of all matrices of an arbitrary type with entries in a field $\mathbb{F}$, i.e.,

$$M(\mathbb{F}) = \bigcup_{m,n\in\mathbb{N}} \mathbb{F}^{m\times n},$$

let $\varnothing$ be an element which is not a member of $M(\mathbb{F})$, and let $M_\varnothing(\mathbb{F}) = M(\mathbb{F}) \cup \{\varnothing\}$. Define the multiplication in $M_\varnothing(\mathbb{F})$ such that the product in $M_\varnothing(\mathbb{F})$ of two matrices coincides with their ordinary matrix product, if it is defined, and all other products are equal to $\varnothing$. With respect to such multiplication, $M_\varnothing(\mathbb{F})$ is a semigroup with the zero $\varnothing$. We will call $M_\varnothing(\mathbb{F})$ the *semigroup of matrices* with entries in $\mathbb{F}$, and for the sake of convenience, we will call $\varnothing$ the *empty matrix*. Note that the above definition of the multiplication in $M_\varnothing(\mathbb{F})$ is the standard procedure of the theory of semigroups for converting a partial semigroup into a full semigroup. Subsemigroups of the semigroup of matrices are called *matrix semigroups*.

The *range* $\mathscr{R}(A)$ and the *null space* $\mathscr{N}(A)$ of a matrix $A \in \mathbb{F}^{m\times n} \subset M_\varnothing(\mathbb{F})$ are defined by

$$\mathscr{R}(A) = \{\boldsymbol{v} \in \mathbb{F}^m \,|\, (\exists \boldsymbol{u} \in \mathbb{F}^n)\, A\boldsymbol{u} = \boldsymbol{v}\,\}, \quad \mathscr{N}(A) = \{\boldsymbol{u} \in \mathbb{F}^n \,|\, A\boldsymbol{u} = \boldsymbol{0}\,\},$$

and rank(A) denotes the *rank* of A, i.e., the dimension of $\mathscr{R}(A)$ (cf. [3,66]).

It is easy to see that $D_\varnothing = R_\varnothing = L_\varnothing = H_\varnothing = \{\varnothing\}$. On the other hand, for any $A, B \in M(\mathbb{F})$ we have

$$\begin{aligned} A\,\mathscr{R}\,B &\;\Leftrightarrow\; \mathscr{R}(A) = \mathscr{R}(B), \\ A\,\mathscr{L}\,B &\;\Leftrightarrow\; \mathscr{N}(A) = \mathscr{N}(B), \\ A\,\mathscr{H}\,B &\;\Leftrightarrow\; \mathscr{R}(A) = \mathscr{R}(B) \;\wedge\; \mathscr{N}(A) = \mathscr{N}(B), \\ A\,\mathscr{D}\,B &\;\Leftrightarrow\; A\,\mathscr{J}\,B \;\Leftrightarrow\; \mathrm{rank}(A) = \mathrm{rank}(B). \end{aligned}$$

If $\mathscr{R}(A) = \mathscr{R}(B)$ then $A \in \mathbb{F}^{m\times n_1}$ and $B \in \mathbb{F}^{m\times n_2}$, for some $m, n_1, n_2 \in \mathbb{N}$, and if $\mathscr{N}(A) = \mathscr{N}(B)$ then $A \in \mathbb{F}^{m_1\times n}$ and $B \in \mathbb{F}^{m_2\times n}$, for some $m_1, m_2, n \in \mathbb{N}$.

Example 2.9. *(Semigroup of matrices over a semiring)* Let $\mathbb{S}$ be a semiring and let $M_\varnothing(\mathbb{S})$ be the *semigroup of matrices* with entries in $\mathbb{S}$, defined in the same way as the semigroup $M_\varnothing(\mathbb{F})$ from Example 2.8.

The *row space* $\mathfrak{R}(A)$ of a matrix $A \in \mathbb{S}^{m\times n}$ is the span (set of all possible linear combinations) of its row vectors, and the *column space* $\mathfrak{C}(A)$ of A is the span of its column vectors. Note that $\mathfrak{R}(A)$ is considered as a subsemimodule of the (left) $\mathbb{S}$-semimodule $\mathbb{S}^n$ ($\mathbb{S}^{1\times n}$), and $\mathfrak{C}(A)$ as a subsemimodule of the (right) $\mathbb{S}$-semimodule $\mathbb{S}^m$ ($\mathbb{S}^{m\times 1}$). The *row rank* of A, denoted by $\varrho_r(A)$, is the smallest possible cardinality of a spanning set for the row space, and the *column rank* of A, denoted by $\varrho_c(A)$, is the smallest possible cardinality of a spanning set for the column space (cf., e.g., [20,41]). Unlike matrices with entries in a field, the row rank and the column rank of a matrix with entries in a semiring are not necessarily equal.

For an arbitrary matrix $A \in \mathbb{S}^{m\times n}$, any $X \in \mathbb{S}^{n\times p}$ induces an epimorphism of $\mathfrak{R}(A)$ onto $\mathfrak{R}(AX)$ sending $\boldsymbol{u} \in \mathfrak{R}(A)$ to $\boldsymbol{u}X$, and any $Y \in \mathbb{S}^{q\times m}$ induces an epimorphism of $\mathfrak{C}(A)$ onto $\mathfrak{C}(YA)$ sending $\boldsymbol{v} \in \mathfrak{C}(A)$ to $Y\boldsymbol{v}$.

As in Example 2.8, we have $D_\varnothing = R_\varnothing = L_\varnothing = H_\varnothing = \{\varnothing\}$, and for arbitrary $A, B \in M(\mathbb{S})$ it follows

$$\begin{aligned} A\,\mathscr{R}\,B &\;\Leftrightarrow\; \mathfrak{C}(A) = \mathfrak{C}(B), \\ A\,\mathscr{L}\,B &\;\Leftrightarrow\; \mathfrak{R}(A) = \mathfrak{R}(B), \\ A\,\mathscr{H}\,B &\;\Leftrightarrow\; \mathfrak{C}(A) = \mathfrak{C}(B) \;\wedge\; \mathfrak{R}(A) = \mathfrak{R}(B), \end{aligned}$$

$$
\begin{aligned}
A \mathscr{D} B \quad &\Leftrightarrow \quad \text{there exist matrices which induce mutually inverse} \\
&\qquad\qquad \text{isomorphisms between } \mathfrak{R}(A) \text{ and } \mathfrak{R}(B) \\
&\Leftrightarrow \quad \text{there exist matrices which induce mutually inverse} \\
&\qquad\qquad \text{isomorphisms between } \mathfrak{C}(A) \text{ and } \mathfrak{C}(B), \\
A \mathscr{J} B \quad &\Leftrightarrow \quad \text{there exist matrices which induce homomorphisms of } \mathfrak{R}(A) \\
&\qquad\qquad \text{into } \mathfrak{R}(B), \text{ and vice versa} \\
&\Leftrightarrow \quad \text{there exist matrices which induce homomorphisms of } \mathfrak{C}(A) \\
&\qquad\qquad \text{into } \mathfrak{C}(B), \text{ and vice versa,}
\end{aligned}
$$

(cf. [41]). Note that $\mathfrak{C}(A) = \mathfrak{C}(B)$ implies $A \in \mathbb{S}^{m\times n_1}$ and $B \in \mathbb{S}^{m\times n_2}$, for some $m, n_1, n_2 \in \mathbb{N}$, whereas $\mathfrak{R}(A) = \mathfrak{R}(B)$ implies $A \in \mathbb{S}^{m_1\times n}$ and $B \in \mathbb{S}^{m_2\times n}$, for some $m_1, m_2, n \in \mathbb{N}$.

Next we proceed with generalized inverses. First we recall that an *involutive semigroup* is a semigroup S equipped with a unary operation * (called *involution*) satisfying $(ab)^* = b^*a^*$ and $(a^*)^* = a$, for all $a, b \in S$. Depending on the context, in the sequel S will denote a semigroup or an involutive semigroup (in the cases when the equations (3) and (4) given below are taken into account). Let us consider the equations

(1) $axa = a$,
(2) $xax = x$,
(3) $(ax)^* = ax$,
(4) $(xa)^* = xa$,
(5) $ax = xa$,

where $a \in S$ is a given element, and x is an unknown taking values in S. Let us note that equations (3) and (4), as well as involutive semigroups, will not be considered in the rest of this paper. However, in the definitions we give here, we include these two equations to emphasize the significance of equations (1) and (2) and their connection with fundamental types of generalized inverses – Moore-Penrose inverses, minimum-norm g-inverses and least-squares g-inverses (see the definitions below). This also applies to equation (5), which is very important for recently intensively studied core and dual core inverses.

For any $\gamma \subseteq \{1,2,3,4,5\}$, the system consisting of the equations (i), for $i \in \gamma$, is denoted by (γ), and solutions to (γ) are called *γ-inverses* of a. The set of all γ-inverses of a will be denoted by $a\gamma$, and the set of all γ-inverses of a contained in a set X will be denoted by $a\gamma_X$.

Commonly, a $\{1\}$-inverse is called a *g-inverse*, a $\{2\}$-inverse is called an *outer inverse*, and a $\{1,2\}$-inverse is called a *reflexive inverse*. A $\{1,3\}$-inverse is known as a *least-squares inverse*, a $\{1,4\}$-inverse is known as a *minimum-norm inverse*, a $\{1,2,3,4\}$-inverse is known as the *Moore-Penrose inverse* or shortly *MP-inverse* of a, and a $\{1,2,5\}$-inverse is known as a *group inverse* of a. If a has at least one γ-inverse, then it is said to be *γ-invertible*. It is worth noting that an element of a semigroup having a $\{1\}$-inverse is called a *regular element*, and a semigroup whose every element is regular is called a *regular semigroup*.

Examples of regular semigroups are the semigroup of matrices with entries in a field, the full transformation semigroup and the semigroup of linear transformations. In fact, the regularity of the full transformation semigroup of an infinite set and the semigroup of linear transformations of an infinitely dimensional vector space is a consequence of the Axiom of Choice.

For the sake of simplicity, the set of all {1}-invertible elements of a semigroup S will be denoted by $S^{(1)}$ (this should not be confused with the notation $S^{\mathbb{1}}$ introduced at the beginning of Section 2). In the theory of semigroups, this set is usually denoted by $Reg(S)$, but here the first notation is more convenient. Similarly, the set of all {2}-invertible elements of S will be denoted by $S^{(2)}$. In the case of the existence, the Moore-Penrose inverse and the group inverse of an element a are unique, and they are denoted by $a^{\dagger}$ and $a^{\#}$, respectively. The set of all idempotents contained in a subset X of a semigroup S will be denoted by $X^{\bullet}$.

It is worth noting that if T is a regular subsemigroup of a semigroup S, then any of Green's equivalences on T is simply the restriction of the corresponding Green's equivalence on S to $T \times T$. In particular, if T is a regular subsemigroup of a full transformation semigroup, semigroup of linear transformations or a semigroup of matrices, then Green's equivalences $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ on T can be characterized in the same way as in Examples 2.6, 2.7 and 2.8, in terms of images/ranges and kernels/null spaces.

3. Outer and inner inverses belonging to the prescribed Green's $\mathscr{R}$- and $\mathscr{L}$-classes and one-sided principal ideals

In this section, we consider outer inverses of a given element belonging to the prescribed Green's $\mathscr{R}$-classes and $\mathscr{L}$-classes, as well as inner inverses belonging to the prescribed right and left principal ideals and reflexive generalized inverses belonging to the prescribed Green's $\mathscr{R}$-classes and $\mathscr{L}$-classes.

First, we state and prove the following theorem that can be viewed as the semigroup-theoretical counterpart of Theorem 1.3.7 [66] (called there the Urquhart formula; see also [63]). This theorem is fundamental as it provides, as we will see, both an existence criterion and a formula for calculating (b, c)-inverse.

Theorem 3.1. *Let S be a semigroup and let $a, b, c \in S$ such that $cab \in S^{(1)}$, and let*

$$x = b(cab)^{(1)}c,$$

where $(cab)^{(1)} \in cab\{1\}$ is an arbitrary element. Then the following statements are true:

(a) *$x \in a\{1\}$ if and only if $ab \in R_a$ and $ca \in L_a$;*
(b) *$x \in a\{2\}_{R_b}$ if and only if $cab \in L_b$;*
(b′) *$x \in a\{2\}_{L_c}$ if and only if $cab \in R_c$;*
(c) *$x \in a\{2\}_{R_b \cap L_c}$ if and only if $cab \in R_c \cap L_b$;*
(d) *$x \in a\{1, 2\}_{R_b \cap L_c}$ if and only if ab and ca are trace products.*

Proof. (a) Let $x \in a\{1\}$, i.e., $a = axa$. Then

$$a = axa = ab(cab)^{(1)}ca \in R(ab) \cap L(ca),$$

and trivially, $ab \in R(a)$ and $ca \in L(a)$, so we obtain that $ab \mathscr{R} a$ and $ca \mathscr{L} a$, that is, $ab \in R_a$ and $ca \in L_a$.

Conversely, let $ab \in R_a$ and $ca \in L_a$. By $ab \mathscr{R} a$ it follows that $a = abs$, for some $s \in S^{\mathbb{1}}$. Since $\mathscr{L}$ is a right congruence and $ca \mathscr{L} a$, it follows $cab \mathscr{L} ab$. On the other hand, $(cab)^{(1)}cab \mathscr{L} cab$ implies $(cab)^{(1)}cab \mathscr{L} ab$, i.e., $(cab)^{(1)}cab \in L_{ab}$. Now it can be concluded $(cab)^{(1)}cab$ is a right identity in L_{ab}, whence

$$ab(cab)^{(1)}cab = ab.$$

Therefore,

$$axa = axabs = ab(cab)^{(1)}cabs = abs = a,$$

and we have proved that $x \in a\{1\}$.

(b) Let $xax = x$ satisfy $x \in R_b$. Then $b = xs$, for some $s \in S^{\mathbb{1}}$, whence

$$b = xs = xaxs = xab = b(cab)^{(1)}cab \in L(cab),$$

and since $cab \in L(b)$ is trivially true, we conclude $cab \mathscr{L} b$, that is, $cab \in L_b$.

Conversely, let $cab \in L_b$. Since $(cab)^{(1)}cab \mathscr{L} cab$, we obtain that $(cab)^{(1)}cab \in L_b$, so $(cab)^{(1)}cab$ is a right identity in L_b, and this yields

$$b(cab)^{(1)}cab = b.$$

Now it follows

$$xax = b(cab)^{(1)}cab(cab)^{(1)}c = b(cab)^{(1)}c = x,$$

as well as

$$x = b(cab)^{(1)}c \in R(b), \qquad b = b(cab)^{(1)}cab = xab \in R(x),$$

whence $x \mathscr{R} b$, i.e., $x \in R_b$.

According to the principle of duality, (b′) is also valid, whereas (c) follows immediately from (b) and (b′).

(d) Let $x \in a\{1,2\}$ and $x \in R_b \cap L_c$. Then by (c) and (a) it follows that $cab \in R_c \cap L_b$, $ab \in R_a$ and $ca \in L_a$. By $ab \mathscr{R} a$ and $ca \mathscr{L} a$ we obtain $cab \mathscr{R} ca$ and $cab \mathscr{L} ab$, whence $ca \mathscr{R} c$ and $ab \mathscr{L} b$. Thus, $ab \in R_a \cap L_b$ and $ca \in R_c \cap L_a$, i.e., ab and ca are trace products.

Conversely, let $ab \in R_a \cap L_b$ and $ca \in R_c \cap L_a$. Then by $ab \mathscr{R} a$ and $ca \mathscr{L} a$ it follows $cab \mathscr{R} ca \mathscr{R} c$ and $cab \mathscr{L} ab \mathscr{L} b$, and hence, $cab \in R_c \cap L_b$. Now, the statements (a) and (c) imply $x \in a\{1,2\}$. □

The situation considered in Theorem 3.1 (d) is shown in Figure 2.

The assertion of Corollary 3.2 is obtained directly from Theorem 3.1, omitting the element c. Its dual assertion, presented in Corollary 3.3, is obtained also from Theorem 3.1, omitting the element b.

Corollary 3.2. *Let S be a semigroup and let $a, b \in S$ such that $ab \in S^{(1)}$, and let*

$$x = b(ab)^{(1)}.$$

Then the following statements are true:

	L_b		L_a		L_c	
R_b	b		xa		x	
R_a	ab		a		ax	
R_c	cab		ca		c	

FIGURE 2. Visualisation of the situation considered in Theorem 3.1 (d)

(a) *$x \in a\{1\}$ if and only if $ab \in R_a$;*
(b) *$x \in a\{2\}_{R_b}$ if and only if $ab \in L_b$;*
(c) *$x \in a\{1,2\}_{R_b}$ if and only if ab is a trace product.*

Corollary 3.3. *Let S be a semigroup and let $c, a \in S$ be such that $ca \in S^{(1)}$, and let*

$$x = (ca)^{(1)}c.$$

Then the following statements are true:

(a) *$x \in a\{1\}$ if and only if $ca \in L_a$;*
(b) *$x \in a\{2\}_{L_c}$ if and only if $ca \in R_c$;*
(c) *$x \in a\{1,2\}_{L_c}$ if and only if ca is a trace product.*

Next we provide the existence conditions and characterizations of outer inverses of a given element contained in the prescribed $\mathscr{R}$-class.

Theorem 3.4. *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_b;*
(ii) *there exists an inner inverse of b contained in the principal left ideal $L(a)$;*
(iii) *there exists $x \in R(b)$ such that $b = xab$;*
(iv) *there exists $u \in S$ such that $b = buab$;*
(v) *$ab \in S^{(1)}$ and $ab \in L_b$;*
(vi) *$ab \in S^{(1)}$ and $b(ab)^{(1)}ab = b$, for some (eqivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these assertions are true, then

$$a\{2\}_{R_b} = \{x \in R(b) \mid b = xab\} = \{bu \mid u \in S^{\mathbb{1}} \text{ such that } b = buab\} \qquad (1)$$
$$= \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\},$$
$$b\{1\}_{L(a)} = \{ua \mid u \in S^{\mathbb{1}} \text{ such that } b = buab\}. \qquad (2)$$

Proof. (i)⇒(iii). Let x be an outer inverse of a contained in R_b. By $x \mathscr{R} b$ it follows that $x = bu$, for some $u \in S^{\mathbb{1}}$, so $x = xax = buax \in bSx$. Moreover,

$x \mathscr{R} b$ and $xa \mathscr{R} x$ yield $xa \mathscr{R} b$. Since xa is an idempotent, by Theorem 2.3 it follows that xa is a left identity for $R_{xa} = R_b$, whence $xab = b$.

(iii)⇒(iv). Let $x \in R(b)$ such that $b = xab$. Then $x = bu$, for some $u \in S^{\mathbb{1}}$. If $u \in S$ then clearly $b = buab$, and if $u = \mathbb{1}$, i.e. $x = b$, then $b = bvab$ for $v = ab$.

(ii)⇒(iv). Let v be an inner inverse of b contained in $L(a)$, i.e., let $b = bvb$ and $v = sa$, for some $s \in S^{\mathbb{1}}$. If $s \in S$, then $b = buab$ for $u = s$, and if $s = \mathbb{1}$, then $b = bab$ and $b = buab$ for $u = ab$.

(iv)⇒(ii). If $u \in S$ such that $b = buab$, then $ua \in b\{1\} \cap L(a)$.

(iv)⇒(v). If (iv) holds, then $b \in L(ab)$, and seeing that $ab \in L(b)$, one concludes $ab \mathscr{L} b$, i.e., $ab \in L_b$. Moreover, by (iv) it follows that ab is regular.

(v)⇒(vi). By (v) it follows that $ab \in D_b$, and since b is regular, then ab is also regular. Moreover, for each $(ab)^{(1)} \in ab\{1\}$ we have $(ab)^{(1)}ab \mathscr{L} ab$, and since $ab \mathscr{L} b$, then $(ab)^{(1)}ab \mathscr{L} b$, i.e., $(ab)^{(1)}ab \in L_b$. Seeing that $(ab)^{(1)}ab$ is an idempotent, it is a right identity for L_b, so $b(ab)^{(1)}ab = b$.

(vi)⇒(i). If (vi) holds, then $ab \in R_b$, and by Corollary 3.2 it follows that $x = b(ab)^{(1)}$ is an outer inverse of a contained in R_b.

Therefore, the statements (i)–(vi) are equivalent. Suppose that they are true. If x is an outer inverse of a contained in R_b, according to the proof of (i)⇒(iii) leads to the conclusion $b = xab$ and $x \in R(b)$. Further, if $x \in R(b)$ such that $b = xab$, then by the proof of (iii)⇒(iv) it follows that $x = bu$, for some $u \in S$, and $b = buab$. Next, if $b = buab$, for some $u \in S$, then $ab = abuab$, which means $u \in ab\{1\}$. Finally, by Corollary 3.2 we obtain $b(ab)^{(1)} \in a\{2\}_{R_b}$, for every $(ab)^{(1)} \in ab\{1\}$. Therefore, the following statement is verified:

$$\begin{aligned} a\{2\}_{R_b} &\subseteq \{x \in R(b) \mid b = xab\} \subseteq \{bu \mid u \in S^{\mathbb{1}} \text{ such that } b = buab\} \\ &\subseteq \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\} \subseteq a\{2\}_{R_b}. \end{aligned}$$

Finally, the equation (2) follows directly from the proofs of (ii)⇒(iv) and (iv)⇒(ii).

This completes the proof of the theorem. □

Remark 3.1. Note that

$$\{bu \mid u \in S^{\mathbb{1}} \text{ such that } b = buab\} =$$

$$= \begin{cases} \{bu \mid u \in S \text{ such that } b = buab\} & \text{if } b \neq bab, \\ \{bu \mid u \in S \text{ such that } b = buab\} \cup \{b\} & \text{if } b = bab. \end{cases}$$

Remark 3.2. Theorem 3.4 is a generalization of Theorem 3 [61], which concerns outer inverses of matrices with a prescribed range. There are two main differences between these two theorems.

(a) The assumption $ab \in S^{(1)}$ appearing in Theorem 3.4 is not required in the case of matrices, since the semigroup of matrices is regular. This also holds when dealing with the full transformation semigroup and the semigroup of linear transformations.

(b) One of the equivalent conditions of Theorem 3 [61] is $\mathscr{N}(AB) = \mathscr{N}(B)$, whose semigroup-theoretical counterpart is the condition $ab \mathscr{L} b$ (that is, $ab \in L_b$) appearing in Theorem 3.4. In addition, Theorem 3 [61] contains the condition rank(AB) = rank(B), which is equivalent to $\mathcal{N}(AB) = \mathcal{N}(B)$ (see [3, Chapter 1, Ex. 10] or [66, Theorem 1.1.3]). The condition concerning ranks is critical because in practice it is much easier to check the equality of ranks than the equality of null spaces. However, its semigroup-theoretical counterpart, the condition $ab\,\mathscr{D}\,b$, is not present in Theorem 3.4, because in the general case it is not equivalent to $ab\,\mathscr{L}\,b$. The equivalence of these two conditions is not valid, for example, in the full transformation semigroup of an infinite set and in the semigroup of linear transformations of an infinitely dimensional vector space, but it is true in the full transformation semigroup on a finite set. Namely, if X is a finite set, then for arbitrary $\alpha, \beta \in \mathscr{T}_X$ we have

$$\alpha\beta\,\mathscr{L}\,\beta \;\Leftrightarrow\; \ker(\alpha\beta) = \ker(\beta) \;\Leftrightarrow\; \mathrm{rank}(\alpha\beta) = \mathrm{rank}(\beta) \;\Leftrightarrow\; \alpha\beta\,\mathscr{D}\,\beta, \tag{3}$$

$$\alpha\beta\,\mathscr{R}\,\alpha \;\Leftrightarrow\; \mathrm{im}(\alpha\beta) = \mathrm{im}(\alpha) \;\Leftrightarrow\; \mathrm{rank}(\alpha\beta) = \mathrm{rank}(\alpha) \;\Leftrightarrow\; \alpha\beta\,\mathscr{D}\,\alpha. \tag{4}$$

Example 3.5. Let us consider the full transformation semigroup $\mathscr{T}_X$ on the set $X = \{1, 2, 3, 4\}$, and maps $\alpha, \beta \in \mathscr{T}_X$ given by $\alpha = (1242)$ and $\beta = (1131)$.

Based on $\alpha\beta = (1141)$ so rank($\alpha\beta$) = rank(β), and according to (3) and Theorem 3.4 it ia obtained $\alpha\{2\}_{R_\beta} \neq \emptyset$. For an arbitrary $v \in \mathscr{T}_X$, the equality $\beta v\alpha\beta = \beta$ holds if and only if $\beta v(\alpha(i)) = i$, for each $i \in \mathrm{im}(\beta) = \{1, 3\}$, which means $\beta v\alpha\beta = \beta$ if and only if there are $i, j \in \{1, 3\}$ such that $\beta v = (1ij3)$ (since rank(βv) = rank(β) = 2 must be satisfied).

Therefore, $\alpha\{2\}_{R_\beta} = \{(1113), (1133), (1313), (1333)\}$.

Next, we provide the existence conditions and characterizations of outer inverses of a given element contained in a given $\mathscr{L}$-class. These results generalize Theorem 5 [61].

Theorem 3.6. *Let S be a semigroup and $a, c \in S$. Then the following statements are equivalent:*

- (i) *there exists an outer inverse of a contained in the $\mathscr{L}$-class L_c;*
- (ii) *there exists an inner inverse of c contained in the principal right ideal $R(a)$;*
- (iii) *there exists $x \in L(c)$ such that $c = cax$;*
- (iv) *there exists $v \in S$ such that $c = cavc$;*
- (v) *$ca \in S^{(1)}$ and $ca \in R_c$;*
- (vi) *$ca \in S^{(1)}$ and $ca(ca)^{(1)}c = c$, for some (equivalently every) $(ca)^{(1)} \in ca\{1\}$.*

If these statements are true, then

$$\begin{aligned} a\{2\}_{L_c} &= \{x \in L(c) \mid c = cax\} = \{vc \mid v \in S^{\mathbb{1}} \text{ such that } c = cavc\} \\ &= \{(ca)^{(1)}c \mid (ca)^{(1)} \in ca\{1\}\}, \end{aligned} \tag{5}$$

$$c\{1\}_{R(a)} = \{av \mid v \in S^{\mathbb{1}} \text{ such that } c = cavc\}. \tag{6}$$

Example 3.7. Consider again the full transformation semigroup $\mathscr{T}_X$ on the set $X = \{1, 2, 3, 4\}$, a map $\alpha \in \mathscr{T}_X$ given by $\alpha = (1242)$ (already considered in Example 3.5), and a map $\gamma \in \mathscr{T}_X$ given by $\gamma = (2424)$.

We have $\gamma\alpha = (2444)$ and rank($\gamma\alpha$) = rank(γ), so (4) and Theorem 3.6 yield $\alpha\{2\}_{L_\gamma} \neq \emptyset$. For any $v \in \mathscr{T}_X$ the relation $\gamma\alpha v\gamma = \gamma$ is valid if and only if $\alpha(v\gamma(i)) \in [i]_{\ker(\gamma)}$, for each $i \in X$, where $[i]_{\ker(\gamma)}$ denotes the equivalence class of i with respect to $\ker(\gamma)$. Having in mind that the equivalence classes of $\ker(\gamma)$ are $\{1,3\}$ and $\{2,4\}$ and that $\mathrm{rank}(\upsilon\gamma) = \mathrm{rank}(\gamma) = 2$ must be satisfied, we conclude $\gamma\alpha\upsilon\gamma = \gamma$ if and only if there is $i \in \{2,3,4\}$ such that $\upsilon\gamma = (1i1i)$.

Thus, $\alpha\{2\}_{L_\gamma} = \{(1212), (1313), (1414)\}$.

When working with inner inverses, it is interesting to locate them in prescribed principal right and left ideals, as seen in the next two theorems. These two theorems generalize Theorems 8 and 9 [61], and the same notes are valid as those given in Remark 3.2.

Theorem 3.8. *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(i) *there exists an inner inverse of a contained in the principal right ideal $R(b)$;*
(ii) *there exists a $\{1,2\}$-inverse of a contained in the principal right ideal $R(b)$;*
(iii) *there exists $u \in S$ such that $a = abua$;*
(iv) *$ab \in S^{(1)}$ and $ab \in R_a$;*
(v) *$ab \in S^{(1)}$ and $ab(ab)^{(1)}a = a$, for some (equivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these equivalences are true, then

$$a\{1\}_{R(b)} = \{bu \mid u \in S^{\mathbb{1}} \text{ such that } a = abua\} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\}, \tag{7}$$

$$a\{1,2\}_{R(b)} = \{xax \mid x \in a\{1\}_{R(b)}\} = \{buabu \mid u \in S^{\mathbb{1}} \text{ such that } a = abua\}. \tag{8}$$

Proof. (i)⇒(ii). If $x \in a\{1\}$ and $x \in R(b)$, then $xax \in a\{1,2\}$ and $xax \in R(b)$. The implication (ii)⇒(i) is straightforward.

(i)⇒(iii). Let $x \in a\{1\}$ and $x \in R(b)$. In this case, $x = bs$ for some $s \in S^{\mathbb{1}}$, and for $u = sax$ it can be concluded

$$a = axa = axaxa = absaxa = abua.$$

(iii)⇒(iv). Let $a = abua$, for some $u \in S$. Noting that $ab \in R(a)$, we conclude $ab \,\mathscr{R}\, a$, i.e., $ab \in R_a$. On the other hand, $ab = abuab$, so ab is regular, which also implies regularity of a.

(iv)⇒(v). By (iv) it follows that $ab \in D_a$, and since a is regular, then ab is also regular. Moreover, for each $(ab)^{(1)} \in ab\{1\}$ we have $ab(ab)^{(1)} \,\mathscr{R}\, ab$, and since $ab \,\mathscr{R}\, a$, it can be concluded that $ab(ab)^{(1)} \,\mathscr{R}\, a$, i.e., $ab(ab)^{(1)} \in R_a$. Seeing that $ab(ab)^{(1)}$ is an idempotent, it is a left identity for R_a, so $ab(ab)^{(1)}a = a$.

(v)⇒(i). If (v) holds, then $ab \in R_a$, and according to Corollary 3.2, it follows $x = b(ab)^{(1)} \in a\{1\}$, and also, $x = b(ab)^{(1)} \in R(b)$.

Thus, we have proved that the statements (i)–(v) are equivalent. Assume that they are true. If $x \in a\{1\}$ and $x \in R(b)$, then $x = bu$, for some $u \in S^{\mathbb{1}}$, so $a = axa = abua$. Next, let $x = bu$, for some $u \in S^{\mathbb{1}}$ such that $a = abua$. If $u \in S$, then $u = (ab)^{(1)} \in ab\{1\}$, so $x = b(ab)^{(1)}$, and if $u = \mathbb{1}$, then $a = aba = a(b(ab))a$, and since ab is an idempotent, then we have that $ab \in ab\{1\}$ and $b = b(ab) = b(ab)^{(1)}$. Finally, if $x = b(ab)^{(1)}$, for some $(ab)^{(1)} \in ab\{1\}$, then by (iv) it follows that $x \in a\{1\}$, and clearly, $x \in R(b)$. Thus, it is proved that

$$a\{1\}_{R(b)} \subseteq \{bu \mid u \in S^{\mathbb{1}} \text{ such that } a = abua\} \subseteq \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\} \subseteq a\{1\}_{R(b)}.$$

Next, for any $x \in a\{1,2\}_{R(b)}$ it is concluded $x \in a\{2\}_{R(b)}$ and $x = xax$. Based on this fact and the proof of (i)⇒(ii) it follows that

$$a\{1,2\}_{R(b)} = \{xax \mid x \in a\{1\}_{R(b)}\}.$$

This completes the proof of the theorem. □

Due to duality, we also have that the following assertion holds true.

Theorem 3.9. *Let S be a semigroup and $a, c \in S$. Then the following statements are equivalent:*

(i) *there exists an inner inverse of a contained in the principal left ideal $L(c)$;*
(ii) *there exists a $\{1,2\}$-inverse of a contained in the principal left ideal $L(c)$;*
(iii) *there exists $v \in S$ such that $a = avca$;*
(iv) *$ca \in S^{(1)}$ and $ca \in L_a$;*
(v) *$ca \in S^{(1)}$ and $a(ca)^{(1)}ca = a$, for some (equivalently every) $(ca)^{(1)} \in ca\{1\}$.*

If these equivalences are true, then

$$a\{1\}_{L(c)} = \{vc \mid v \in S^{\mathbb{1}} \text{ such that } a = avca\} = \{(ca)^{(1)}c \mid (ca)^{(1)} \in ca\{1\}\}, \quad (9)$$
$$a\{1,2\}_{L(c)} = \{xax \mid x \in a\{1\}_{L(c)}\}. \quad (10)$$

Let us recall that an element a of a semigroup S is called *regular* if there exists $x \in S$ such that $a = axa$, i.e., if a has an inner inverse. Besides, a is called *left regular* if there exists $x \in S$ such that $a = xa^2$, and it is called *right regular* if there exists $x \in S$ such that $a = a^2x$. Applying the previously obtained results, we provide the following characterization of both regular and left regular elements.

Theorem 3.10. *Let S be a semigroup and $a \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{R}$-class R_a;*
(ii) *there exists an inner inverse of a contained in the principal left ideal $L(a)$;*
(iii) *there exists a $\{1,2\}$-inverse of a contained in the principal left ideal $L(a)$;*
(iv) *there exists $x \in R(a)$ such that $a = xa^2$;*
(v) *there exists $u \in S$ such that $a = aua^2$;*
(vi) *a is regular and left regular.*

Further, the following representations hold under these equivalences:

$$a\{2\}_{R_a} = \{x \in R(a) \mid a = xa^2\} = \{au \mid u \in S^{\mathbb{1}} \text{ such that } a = aua^2\}, \quad (11)$$
$$a\{1\}_{L(a)} = \{ua \mid u \in S^{\mathbb{1}} \text{ such that } a = aua^2\} \quad (12)$$

Proof. The equivalence of statements (i), (iv) and (v) is obtained by applying Theorem 3.4 with $b = a$, while the equivalence of statements (ii), (iii) and (v) is obtained by applying Theorem 3.9 with $c = a$. The equivalence of (v) and (vi) is straightforward. Besides, (11) one obtains by putting $b = a$ in (1), whereas (12) one obtains by putting $c = a$ in (9). □

A theorem dual to Theorem 3.10 is also valid.

Theorem 3.11. *Let S be a semigroup and $a \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{L}$-class L_a;*
(i) *there exists an inner inverse of a contained in the principal right ideal $R(a)$;*
(ii) *there exists a $\{1,2\}$-inverse of a contained in the principal right ideal $R(a)$;*
(ii) *there exists $x \in L(a)$ such that $a = a^2x$;*
(iii) *there exists $u \in S$ such that $a = a^2ua$;*
(iv) *a is regular and right regular.*

If these statements are true, then

$$a\{2\}_{L_a} = \{x \in L(a) \mid a = a^2x\} = \{ua \mid u \in S^{\mathbb{1}} \text{ such that } a = a^2ua\}, \tag{13}$$

$$a\{1\}_{R(a)} = \{au \mid u \in S^{\mathbb{1}} \text{ such that } a = a^2ua\}. \tag{14}$$

For more information on the relationship between the equations $a = axa$, $a = xa^2$, $a = axa^2$, $a = a^2x$ and $a = a^2xa$, as well as their place in the implication diagram of all regularity types of elements of a semigroup defined by linear equations, we refer to [10].

Combining Theorems 3.4 and 3.8 we obtain the following theorem that provides the existence conditions and characterizations of a reflexive g-inverse of a given element contained in a given $\mathscr{R}$-class. This theorem is a generalization of Theorem 10 [61].

Theorem 3.12. *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_b;*
(ii) *there exist $u, v \in S$ such that $b = buab$ and $a = abva$;*
(iii) *there exists $w \in S$ such that $b = bwab$ and $a = abwa$;*
(iv) *there exist $s, t \in S$ such that $a = abs$ and $b = tab$;*
(v) *ab is a trace product;*
(vi) *$ab \in S^{(1)}$, $ab(ab)^{(1)}a = a$ and $b(ab)^{(1)}ab = b$, for some (equivalently every) $(ab)^{(1)} \in ab\{1\}$.*

If these statements hold, then

$$a\{1,2\}_{R_b} = a\{2\}_{R_b} = a\{1\}_{R(b)}. \tag{15}$$

Proof. (i)⇒(ii). This follows directly by Theorems 3.4 and 3.8, since $R_b \subseteq R(b)$.

(ii)⇒(iii). Let $b = buab$ and $a = abva$, for some $u, v \in S$. Then

$$b = buab = bu(abva)b = (buab)vab = bvab,$$

$$a = abva = a(buab)va = abu(abva) = abua,$$

and hence, (iii) holds.

(iii)⇒(iv). This implication is straightforward.

(iv)⇒(v). The condition (iv) means that $a \in R(ab)$ and $b \in L(ab)$, and since $ab \in R(a) \cap L(b)$ always holds, we conclude that $ab \in R_a \cap L_b$, that is, ab is a trace product.

(v)⇒(vi). Let ab be a trace product, i.e., $ab \in R_a \cap L_b$. By the Miller-Clifford's theorem, the $\mathscr{H}$-class $R_b \cap L_c$ contains an idempotent, which implies that the $\mathscr{D}$-class containing a, b and ab is regular, and therefore, R_a and L_b are regular. Now, by Theorems 3.4 and 3.8 we obtain that (vi) holds.

(vi)⇒(i). If (vi) holds, then by Theorems 3.4 and 3.8 $x = b(ab)^{(1)}$ is a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_b.

Suppose now that the equivalent statements (i)–(vi) hold. Then the equivalent statements of Theorems 3.4 and 3.8 also hold, so b is regular, $R(b) = bS$ and

$$a\{2\}_{R_b} = \{b(ab)^{(1)} \mid (ab)^{(1)} \in ab\{1\}\} = a\{1\}_{bS} = a\{1\}_{R(b)}.$$

Besides, if $x \in a\{2\}_{R_b} = a\{1\}_{R(b)}$ then $x \in a\{1\}$, whence $x \in a\{1,2\}_{R_b}$, and therefore, $a\{2\}_{R_b} \subseteq a\{1,2\}_{R_b}$. Since the opposite inclusion is clear, we conclude that (15) is fulfilled. □

The equivalence (i)⇔(v) has been proven in a different form by Xu and Benítez [75, Theorem 3.14] in the context of rings.

Remark 3.3. According to the previous theorem, a necessary and sufficient condition for the existence of a $\{1,2\}$-inverse of the element a contained in the $\mathscr{R}$-class R_b is the solvability of both equations $b = buab$ and $a = abva$, where u and v are unknowns taking values in the semigroup S. However, when both equations are solvable, according to the proof of the implication (ii)⇒(iii) of the previous theorem, both equations have the same set of solutions. Therefore, if any of the equivalent conditions of Theorem 3.12 is satisfied, to compute a $\{1,2\}$-inverse of a contained in R_b it is enough to find a solution only to one of these two equations.

Due to duality, the following assertion is also true.

Theorem 3.13. *Let S be a semigroup and $a, c \in S$. Then the following statements are equivalent:*

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{L}$-class L_c;*
(ii) *there exist $u, v \in S$ such that $c = cauc$ and $a = avca$;*
(iii) *there exists $w \in S$ such that $c = cawc$ and $a = awca$;*
(iv) *there exist $s, t \in S$ such that $a = cas$ and $c = tca$;*
(v) *ca is a trace product;*
(vi) *$ca \in S^{(1)}$, $ca(ca)^{(1)}c = c$ and $a(ca)^{(1)}ca = a$, for some (every) $(ca)^{(1)} \in ca\{1\}$.*

If these equivalences are true, then

$$a\{1,2\}_{L_c} = a\{2\}_{L_c} = a\{1\}_{L(c)}. \tag{16}$$

We end this section with the following example.

Example 3.14. (Example in a semigroup of square matrices). Consider matrices $A, B \in \mathbb{C}^{3\times 3}$ given by

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -1 & 5 \\ 1 & 7 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}.$$

According to Theorem 2.3, the matrix equation $BUAB = B$ is solvable, since $\operatorname{rank}(AB) = \operatorname{rank}(B) = 2$, which yields $\mathscr{N}(AB) = \mathscr{N}(B)$. One possible solution of $BUAB = B$ is

$$U = (AB)^{\dagger} = \begin{bmatrix} 356/4947 & -25/1649 & -40/1649 \\ -96/8245 & -128/8245 & 25/1649 \\ 192/8245 & 256/8245 & -50/1649 \end{bmatrix}$$

and the outer inverse X which fulfills $\mathscr{R}(X) = \mathscr{R}(B)$ is given by

$$X = BU = \begin{bmatrix} 452/1649 & 53/1649 & -245/1649 \\ -96/1649 & -128/1649 & 125/1649 \\ -260/1649 & 203/1649 & -5/1649 \end{bmatrix}.$$

In contrast, the equation $ABVA = A$ is not solvable, since $\operatorname{rank}(AB) = 2 \neq \operatorname{rank}(A) = 3$, so $\mathscr{R}(AB) = \mathscr{R}(A)$ does not hold. Therefore, there is a $\{2\}$-inverse of A with the range $\mathscr{R}(B)$, i.e., in the $\mathscr{R}$-class of B, but there is no a $\{1,2\}$-inverse nor a $\{1\}$-inverse of A with this range.

4. Outer and inner inverses belonging to the prescribed Green's $\mathscr{H}$-classes

Let S be a semigroup and $a, b, c \in S$. According to Drazin [24], an element $x \in S$ is called a (b,c)-*inverse* of a if it satisfies

(D1) $x \in bSx \cap xSc$;
(D2) $xab = b$ and $cax = c$.

If a has a (b,c)-inverse, we say that it is (b,c)-*invertible*. It can be easily verified that, if the condition (D2) is fulfilled, the condition (D1) can be replaced by a simpler condition $x \in bS \cap Sc$, or by a condition

(D1′) $x \in R(b) \cap L(c)$.

Drazin in [24] proved some basic properties of (b,c)-inverses, including the fact that the (b,c)-inverse of a is its outer inverse. Here we provide a more precise characterization of (b,c)-inverses.

Theorem 4.1. *Let S be a semigroup and $a, b, c \in S$. An element $x \in S$ is a (b,c)-inverse of a if and only if it is an outer inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$.*

Proof. This follows immediately from Theorems 3.4 and 3.6. □

A very important consequence of Theorem 4.1 is the following one:

Corollary 4.2. *Let S be a semigroup and $a, b, c, d, x \in S$. Then*

(a) *x is a (b,c)-inverse of a if an only if it is a (u,v)-inverse of a for some (equivalently all) $u \in R_b$, $v \in L_c$;*
(b) *x is a (b,c)-inverse of a if and only if it is a (d,d)-inverse of a for some (equivalently every) $d \in R_b \cap L_c$.*
(c) *x is a (d,d)-inverse of a if and only if it is an outer inverse of a contained in the $\mathscr{H}$-class H_d.*

In view of Theorem 4.1 and Example 2.8, Drazin's (b,c)-inverse in a semigroup is an immediate generalization of the outer inverse of a matrix with a prescribed range and null space. Some related concepts have also been discussed in the literature on generalized inverses. In the sequel, we show that Drazin's (b,c)-inverse, Mary's inverse along an element and Bott and Duffin's (e,f)-inverse are essentially the same concepts, and that Djordjević and Wei's outer inverse with prescribed idempotents is their particular proper case. The equivalence of Mary's inverse along an element, Drazin's (b,c)-inverse and Bott and Duffin's (e,f)-inverse has been established recently in [48, Theorem C.4].

Remark 4.1 (*Mary's inverse along an element).* For two elements a and d of a semigroup S, Mary in [44] defined an *inverse of a along d* as an outer inverse of a contained in the $\mathscr{H}$-class H_d. Mary also pointed out that an inverse of a along an element d can equivalently be defined as an element $x \in S$ which satisfies the following two conditions:

(M1) $x \in R(d) \cap L(d)$;
(M2) $xad = d$ and $dax = d$.

According to Theorem 4.1 and its corollaries, Drazin's (b,c)-inverse is essentially the same as Mary's inverse along an element d of a semigroup. Namely, according to Corollary 4.2 (b) and (c), Mary's inverse of a along d is Drazin's (b,c)-inverse of a for every pair $b,c \in S$ such that $H_d = R_b \cap L_c$. In particular, we can assume that $(b,c) = (d,d)$. On the other hand, by Corollary 4.2 (b), Drazin's (b,c)-inverse of a is Mary's inverse of a along d, for every $d \in R_b \cap L_c$. Thus, the only difference between Drazin's and Mary's inverses lies in representing $\mathscr{H}$-classes. In the first case, a $\mathscr{H}$-class is given as the intersection of the $\mathscr{R}$-class represented by b and the $\mathscr{L}$-class represented by c. In the second case, the same $\mathscr{H}$-class is represented by its element d.

Remark 4.2 (*Bott-Duffin (e,f)-inverse).* In Drazin's formulation given in [24] (see also [11]), for $a \in S$ and idempotents $e,f \in S^\bullet$, an element $x \in S$ is a *Bott-Duffin (e,f)-inverse* of a if

$$x = ex = xf, \quad xae = e, \quad fax = f. \tag{1}$$

Drazin's (b,c)-inverse is essentially the same as Bott-Duffin (e,f)-inverse corresponding to a pair (e,f) of idempotents of a semigroup S. As noted by Drazin in [24], Bott-Duffin (e,f)-inverse is Drazin's (b,c)-inverse for $(b,c) = (e,f)$. Moreover, and if x is Drazin's (b,c)-inverse of a, then it is also Bott-Duffin (xa,ax)-inverse of a. In fact, Drazin's (b,c)-inverse is Bott-Duffin (e,f)-inverse for any $e \in R_b$ and $f \in L_c$, and in particular, it is Bott-Duffin $(bb^{(1)}, c^{(1)}c)$-inverse for every $b^{(1)} \in b\{1\}$ and $a^{(1)} \in a\{1\}$. Note that $b^{(1)}$ and $a^{(1)}$ exist because each element of the $\mathscr{D}$-class containing b, c, cab and x is regular, whenever there is a (b,c)-inverse x of a.

Remark 4.3 (*Djordjević and Wei's outer inverse with prescribed idempotents).* For a ring R with identity 1, $a \in R$ and $p,q \in R^\bullet$, Djordjević and Wei [23] defined a (p, q)-*outer inverse* of a as an element $x \in R$ satisfying

$$x = xax, \quad xa = p, \quad ax = 1 - q. \tag{2}$$

We also call x an *outer inverse of a with prescribed idempotents* p and $1 - q$. This concept can be easily transmitted into the context of semigroups, replacing the pair $(p, 1-q)$ by a pair (e, f) of idempotents of a semigroup S. If there is an outer inverse of $a \in S$ with prescribed idempotents e and f, it is unique (cf. [23]).

Unlike Mary's concept of an inverse along an element, Bott-Duffin's concept of an (e, f)-inverse, and Drazin's concept of a (b, c)-inverse, which are mutually equivalent, Djordjević and Wei's outer inverses with prescribed idempotents form their proper special case. Namely, if x is an outer inverse of a with prescribed idempotents e and f, then it is easy to check that x is Bott-Duffin (e, f)-inverse of a, and hence, it is Drazin's (e, f)-inverse of a. The converse does not hold. As Example 4.3 shows, if x is Drazin's (e, f)-inverse of a, for some idempotents $e, f \in S^{\bullet}$, then it is not necessary an outer inverse of a with prescribed idempotents e and f.

Example 4.3. Consider the matrix $A \in \mathbb{C}^{3\times 3}$ and its Moore-Penrose inverse $X = A^{\dagger}$, which are given by

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -2 \\ -3 & -1 & 2 \end{bmatrix}, \qquad X = A^{\dagger} = \begin{bmatrix} 1/6 & 0 & -1/6 \\ -1/15 & 1/15 & -1/15 \\ 2/15 & -2/15 & 2/15 \end{bmatrix}.$$

Then the projectors AX and XA are given by

$$XA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & -2/5 \\ 0 & -2/5 & 4/5 \end{bmatrix}, \qquad AX = \begin{bmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{bmatrix}.$$

Further, consider idempotent matrices E and F represented by

$$E = \begin{bmatrix} 1 & -1/15 & -1/30 \\ 0 & 17/75 & -29/75 \\ 0 & -34/75 & 58/75 \end{bmatrix}, \qquad F = \begin{bmatrix} 31/36 & -1/3 & -7/36 \\ -5/18 & 1/3 & -7/18 \\ -5/36 & -1/3 & 29/36 \end{bmatrix}.$$

It is easy to verify that $XAE = E$ and $EXA = XA$, so $E \mathscr{R} XA \mathscr{R} X$, and $AXF = AX$ and $FAX = F$, whence $F \mathscr{L} AX \mathscr{L} X$. Thus, $X \mathscr{R} E$ and $X \mathscr{L} F$, so X is the Drazin's (E, F)-inverse of A. In contrast, X differs from the outer inverse of A with prescribed idempotents E and F, since $E \neq XA$ and $F \neq AX$.

It is known that Drazin's (b, c)-inverse and Mary's inverse along an element are unique, whenever they exist. The next theorem shows that their uniqueness can be derived from the well-known properties of classes of Green's equivalences stated in Green's theorem and lemmas.

Theorem 4.4. *An element of a semigroup can have at most one outer inverse contained in a given* $\mathscr{H}$*-class.*

Proof. Let S be a semigroup and $a \in S$. Suppose that $x, y \in S$ are outer inverses of a such that $x \mathscr{H} y$. Since $\mathscr{L}$ is a right congruence, $xa \mathscr{L} ya$ is implied by $x \mathscr{L} y$. On the other hand, $x \mathscr{R} y$, $x \mathscr{R} xa$ and $y \mathscr{R} ya$ yield $xa \mathscr{R} ya$. Thus, $xa \mathscr{H} ya$, and since both xa and ya are idempotents, we conclude $xa = ya$. Finally, according to Green's lemma, the mapping $\varrho_a|_{L_x}$ is a bijection of L_x onto L_{xa}, and by $\varrho_a(x) = \varrho_a(y)$ it follows $x = y$. □

Now we state the existence criteria for (b, c)-inverses. Equivalence of statements (i), (ii) and (vi) has been proved by Drazin [24], while (ix), (x) and (xi) come from Mary [44] and Mary and Patrício [49]. For the sake of completeness, we quote all these statements and give different proofs.

Theorem 4.5. *Let S be a semigroup and $a, b, c \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$. i.e., a is (b, c)-invertible;*
(ii) *there exist $u, v \in S$ satisfying $b = vcab$ and $c = cabu$;*
(iii) *there exist $u, v \in S$ satisfying $b = bucab$ and $c = cabvc$;*
(iv) *there exists $w \in S$ satisfying $b = bwcab$ and $c = cabwc$;*
(v) *there exist $u, v \in S$ satisfying $b = buab$, $c = cavc$ and $bu = vc$;*
(vi) *$cab \in R_c \cap L_b$;*
(vii) *$cab \in S^{(1)}$, $cab(cab)^{(1)}c = c$ and $b(cab)^{(1)}cab = b$, for some (equivalently every) $(cab)^{(1)} \in cab\{1\}$;*
(viii) *$ab, ca \in S^{(1)}$, $b(ab)^{(1)}ab = b$, $ca(ca)^{(1)}c = c$ and $b(ab)^{(1)} = (ca)^{(1)}c$, for some $(ab)^{(1)} \in ab\{1\}$ and $(ca)^{(1)} \in ca\{1\}$;*
(ix) *$dad \in H_d$, for some (equivalently every) $d \in R_b \cap L_c$;*
(x) *ad is group invertible and $ad \in L_d$, for some (equivalently every) $d \in R_b \cap L_c$;*
(xi) *da is group invertible and $da \in R_d$, for some (equivalently every) $d \in R_b \cap L_c$.*

Proof. The equivalence of statements (i) and (ii) was proved by Drazin in [24] (see also [25]), and (vi) is clearly just another way to write (ii). However, we will give an immediate, illustrative proof for (vi)⇒(i).

(vi)⇒(i). Let $cab \in R_c \cap L_b$. Then $b = ucab$ and $c = cabv$, for some $u, v \in S^{\mathbb{1}}$, and by Green's lemmas it follows that $\lambda_u|_{R_c}$ is a bijective mapping of R_c onto R_b preserving $\mathscr{L}$-classes, and $\varrho_v|_{L_b}$ is a bijective mapping of L_b onto L_c preserving $\mathscr{R}$-classes.

Set $x = ucabv$. Then

$$x = bv = \varrho_v(b) = \varrho_v(\lambda_u(cab)) = \lambda_u(\varrho_v(cab)) = \lambda_u(c) = uc,$$

so $xax = ucabv = x$ and $x \in R_b \cap L_c$ (see Figure 3), and by Theorem 4.1 we obtain that x is a (b, c)-inverse of a.

(ii)⇒(v). If there exist $u, v \in S$ such that $b = vcab$ and $c = cabu$, then $vc = vcabu = bu$, whence

$$b = vcab = buab \qquad \text{and} \qquad c = cabu = cavc.$$

(v)⇒(iv). If there are $u, v \in S$ such that $b = buab$, $c = cavc$ and $bu = vc$, then $bu = buabu$ and $vc = vcavc$, and the following conclusion follows for

$w = uav$:

$$b = buab = buabuab = buavcab = bwcab, \; c = cavc = cavcavc = cabuavc = cabwc.$$

(iv)⇒(iii) and (iii)⇒(ii). These implications are clear.

(ii)⇒(vii). If $b = ucab$ and $c = cabv$, for some $u, v \in S$, then vau is a $\{1\}$-inverse of cab. Moreover, each $(cab)^{(1)} \in cab\{1\}$ satisfies $c = cabv = cab(cab)^{(1)}cabv = cab(cab)^{(1)}c$, and similarly $b = b(cab)^{(1)}cab$.

(vii)⇒(ii). This implication can be simply verified.

The equivalence (i)⇔(ix) is actually a result of Mary and Patricio (Theorem 2.2 [49]), and it can be easily derived as a consequence of the equivalence (i)⇔(vi) from this theorem and of Corollary 4.2. On the other hand, the equivalence of the statements (i), (x) and (xi) has been proved by Mary (Theorem 7 [44]), but here we give a different proof, whose parts will be used in the further work.

(i)⇒(x). Suppose that x is a (b, c)-inverse of a, and consider an arbitrary $d \in R_b \cap L_c$. Clearly, $R_d = R_b = R_x$ and $L_d = L_c = L_x$. Consider $e = ax \in S^\bullet$. By $x \mathscr{R} d$ and the left compatibility of $\mathscr{R}$ it follows that $ax \mathscr{R} ad$. This means that $e = ax$ is an idempotent in R_{ad}, so it is a left identity in R_{ad}, and therefore, $ad = ead$.

Since $d \in R_b$, and $ax \mathscr{L} x$ implies $e = ax \in L_x = L_c$, by Corollary 4.2 we obtain that x is a (d, e)-inverse of a, and by (i)⇔(vi) we conclude that $ad = ead \in R_e \cap L_d = R_e \cap L_c = R_e \cap L_e = H_e$. Thus, ad is group invertible and $ad \in L_d$.

(x)⇒(i). Suppose that ad is group invertible, i.e., $ad \in H_e$, for some $e \in S^\bullet$, and let $ad \in L_d$, for some $d \in R_b \cap L_c$. Then $ead = ad \in R_e \cap L_d$, and by (i)⇔(vi) we obtain that there exists $x \in S$ such that $x = xax$ and $x \in R_d \cap L_e$. However, $R_d = R_b$ and $L_e = L_{ad} = L_d = L_c$, so we conclude that $x \in R_b \cap L_c$.

Analogously we prove (i)⇒(xi) and (xi)⇒(i).

(i)⇒(viii). Suppose that a is (b, c)-invertible, i.e., let there exists $x \in a\{2\}$ such that $x \in R_b \cap L_c$. Then according to Theorem 3.4 and its dual it follows that $ab, ca \in S^{(1)}$, $b(ab)^{(1)}ab = b$, $ca(ca)^{(1)}c = c$, $x = b(ab)^{(1)}$ and $x = (ca)^{(1)}c$, for some $(ab)^{(1)} \in ab\{1\}$ and $(ca)^{(1)} \in ca\{1\}$.

(viii)⇒(v). This implication is straightforward. □

The equivalence of statements (i) and (ix) of the previous theorem can also be stated as follows.

Corollary 4.6. [44] *Let S be a semigroup and $a, d \in S$. Then there exists an outer inverse of a contained in the $\mathscr{H}$-class H_d if and only if $dad \mathscr{H} d$.*

Comments similar to those given in Remark 3.2 can also be given for Theorem 4.5. We will give some more comments concerning this theorem.

Remark 4.4. (a) As we have seen, one of the necessary and sufficient conditions for the existence of a (b, c)-inverse of an element a is the solvability of both equations $b = vcab$ and $c = cabu$, where u and v are unknowns taking values in the semigroup S. However, as noted by Drazin (Remark 2.3 in [24]), if the existence conditions are satisfied, to compute the (b, c)-inverse of a it is

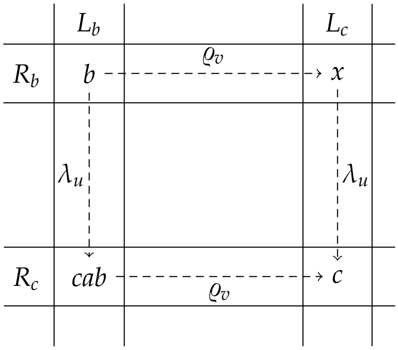


FIGURE 3. Visualisation of the proof of (vi)⇒(i) of Theorem 4.5

enough to solve only one of these two equations. Indeed, if v is an arbitrary solution of the first equation, then the (b,c)-inverse of a is equal to vc, and of u is an arbitrary solution of the second one, then the (b,c)-inverse of a is equal to bu.

(b) Another necessary and sufficient condition for the existence of a (b,c)-inverse of a is the solvability of both equations $b = bucab$ and $c = cabvc$, where u and v are unknowns taking values in S. However, if both equations are solvable, then they have the same set of solutions. Indeed, for random solution u to the first equation and random solution v to the second one we have $buc = bucabvc = bvc$, whence $b = bucab = bvcab$ and $c = cabvc = cabuc$. In this case, for an arbitrary solution u to these two equations, the (b,c)-inverse of a is equal to buc.

(c) It is also worth noting that the conditions $bu = vc$ in (v) and $b(ab)^{(1)} = (ca)^{(1)}c$ in (viii) connect Theorem 3.4 and its dual, in the sense that they provide the existence of an outer inverse of a which belongs both to R_b and L_c. These conditions are necessary because of Example 4.7 given below shows that there are cases in which there exist outer inverses of a in R_b and L_c, but not in $R_b \cap L_c$.

Example 4.7. Consider the full transformation semigroup $\mathscr{T}_X$ on the set $X = \{1,2,3,4\}$, and maps $\alpha,\beta,\gamma \in \mathscr{T}_X$ given by $\alpha = (1242)$, $\beta = (1131)$ and $\gamma = (2442)$.

The following set identities are valid:

$$\alpha\{2\}_{R_\beta} = \{(1333),(1113),(1133),(1313)\},$$
$$\alpha\{2\}_{L_\gamma} = \{(1221),(1441),(3223),(3443)\},$$
$$\alpha\{2\}_{R_\beta\cap L_\gamma} = \emptyset.$$

Therefore, there exist outer inverses of α contained in R_β and L_γ, but there is no an outer inverse of α contained in $R_\beta \cap L_\gamma$.

Example 4.8. Consider the full transformation semigroup $\mathscr{T}_X$ on the set $X = \{1,2,3,4\}$, and maps $\alpha, \beta, \gamma \in \mathscr{T}_X$ given by $\alpha = (1242)$, $\beta = (1311)$ and $\gamma = (2323)$.

Since $\gamma\alpha\beta = (2322)$ implies $\mathrm{rank}(\gamma\alpha\beta) = \mathrm{rank}(\beta) = \mathrm{rank}(\gamma)$, based on (3), (4), Theorem 4.5 and Remark 4.4 (b) it follows that there exists $\lambda, \varrho \in \mathscr{T}_X$ such that $\beta\lambda\gamma\alpha\beta = \beta$ and $\gamma\alpha\beta\varrho\gamma = \gamma$, and then $\beta\lambda\gamma = \beta\varrho\gamma$ and $\alpha\{2\}_{R_\beta \cap L_\gamma} = \{\beta\lambda\gamma\} = \{\beta\varrho\gamma\}$.

By $\beta\lambda\gamma = \beta\varrho\gamma$ it follows that $\beta\lambda\gamma(\alpha(i)) = i$, for each $i \in \mathrm{im}(\beta) = \{1,3\}$, which yields $\beta\lambda\gamma = (1ij3)$, for some $i, j \in \{1,3\}$. On the other hand, by $\gamma\alpha\beta\varrho\gamma = \gamma$ it follows that $\alpha(\beta\varrho\gamma(l)) \in [k]_{\ker(\gamma)}$, for any $k \in X$, whence $\beta\varrho\gamma = (1l1l)$, for some $l \in X$. Hence, we conclude that $i = l = 3$ and $j = 1$, so $\beta\lambda\gamma = \beta\varrho\gamma = (1313)$, i.e., $\alpha\{2\}_{R_\beta \cap L_\gamma} = \{(1313)\}$.

In the same way we obtain that $\alpha\{2\}_{R_\gamma \cap L_\beta} = \{(3233)\}$.

The next theorem provides our first representation of (b,c)-inverses.

Theorem 4.9. *Let S be a semigroup and let elements $a, b, c \in S$ satisfy the equivalent statements of Theorem* 4.5. *Then the unique (b,c)-inverse x of a can be represented as*

$$x = b(cab)^{(1)}c, \tag{3}$$

for any $(cab)^{(1)} \in cab\{1\}$. In addition, x can also be represented by

$$x = b(ab)^{(1)} = (ca)^{(1)}c, \tag{4}$$

for some $(ab)^{(1)} \in ab\{1\}$ and $(ca)^{(1)} \in ca\{1\}$.

Proof. First we note that the expressions $b(cab)^{(1)}c$, $b(ab)^{(1)}$ and $(ca)^{(1)}c$ make sense since cab, ab and ca are regular, according to (iv) and (x) of Theorem 4.5.

By the statement (iv) of Theorem 4.5, for any $(cab)^{(1)} \in cab\{1\}$ we conclude

$$b(cab)^{(1)}cab(cab)^{(1)}c = \left(b(cab)^{(1)}cab\right)(cab)^{(1)}c = b(cab)^{(1)}c,$$

so $b(cab)^{(1)}c$ is an outer inverse of a. In addition, it also holds that

$$b(cab)^{(1)}c \in R(b) \cap L(c),$$

and again by (iv) of Theorem 4.5 we obtain

$$b = b(cab)^{(1)}cab \in R(b(cab)^{(1)}c), \quad c = cab(cab)^{(1)}c \in L(b(cab)^{(1)}c).$$

Thus, $b(cab)^{(1)}c \in R_b \cap L_c$, and according to Theorem 4.1 we have that $b(cab)^{(1)}c$ is a (b,c)-inverse of a.

The representation (4) one obtains directly from Theorem 3.8 and its dual. □

The same representation of the (b,c)-inverse has been given in [75, Theorem 3.5].

The next theorem can be viewed as another formulation of Theorem 7 [44]. However, here we give a different proof, based on the previously proved theorems.

Theorem 4.10. *Let S be a semigroup and $a, b, c \in S$, and let $x \in S$ be a (b, c)-inverse of a. Then for every $d \in R_b \cap L_c$ it follows $ad \in H_{ax}$, $da \in H_{xa}$, and*

$$x = d(ad)^{\#} = (da)^{\#}d. \tag{5}$$

Proof. By the proof of (i)⇒(x) of Theorem 4.5, ad is group invertible, i.e., $ad \in H_e$, for some $e \in S^{\bullet}$, and x is a (d, e)-inverse of a. Since $(ad)^{\#} = (ead)^{\#} \in ead\{1\}$, by Theorem 4.9 it follows

$$x = d(ead)^{\#}e = d(ad)^{\#}e = d(ad)^{\#}.$$

In the same way we prove that $x = (da)^{\#}d$. □

Theorem 4.5 has provided the criteria for an outer inverse belonging to the intersection of an $\mathscr{R}$-class and an $\mathscr{L}$-class. The next theorem does the same for an inner inverse and the intersection of a principal right ideal and a principal left ideal.

Theorem 4.11. *Let S be a semigroup and $a, b, c \in S$. Then the following statements are equivalent:*

(i) *there exists an inner inverse of a contained in the quasi-ideal $R(b) \cap L(c)$;*
(ii) *there exist an inner inverse of a contained in $R(b)$ and an inner inverse of a contained in $L(c)$;*
(iii) *there exist $u, v \in S$ such that $a = abua$ and $a = avca$;*
(iv) *there exists $w \in S$ such that $a = abwca$;*
(v) *$ab, ca \in S^{(1)}$, $ab(ab)^{(1)}a = a$ and $a(ca)^{(1)}ca = a$, for some (equivalently all) $(ab)^{(1)} \in ab\{1\}$ and $(ca)^{(1)} \in ca\{1\}$.*

If these equivalences are true, then

$$\begin{aligned} a\{1\}_{R(b)\cap L(c)} &\supseteq \{yaz \mid y \in a\{1\}_{R(b)}, z \in a\{1\}_{L(c)}\} \\ &\supseteq \{buavc \mid u, v \in S^{\mathbb{1}} \text{ such that } a = abua,\ a = avca\} \\ &= \{b(ab)^{(1)}a(ca)^{(1)}c \mid (ab)^{(1)} \in ab\{1\},\ (ca)^{(1)} \in ca\{1\}\} \\ a\{1\}_{R(b)\cap L(c)} &\supseteq \{bwc \mid w \in S^{\mathbb{1}} \text{ such that } a = abwca\}. \end{aligned}$$

Proof. The equivalence of the statements (ii), (iii) and (v) follows directly by Theorem 3.8, and the implication (i)⇒(ii) is straightforward.

(iii)⇒(iv). Let $u, v \in S$ such that $a = abua$ and $a = avca$. Set $w = uav$. Then

$$abwca = abuavca = avca = a.$$

(iv)⇒(i). Let $w \in S$ such that $a = abwca$. Set $y = bwc$. Then it is clear that $a = aya$ and $y \in R(b) \cap L(c)$.

Let the statements (i)–(v) hold. If $y \in a\{1\}_{R(b)}$, $z \in a\{1\}_{L(c)}$ and $x = yaz$, then it is clear that $a = axa$ and $x \in R(y) \cap L(z) \subseteq R(b) \cap L(c)$, so

$$\{yaz \mid y \in a\{1\}_{R(b)}, z \in a\{1\}_{L(c)}\} \subseteq a\{1\}_{R(b)\cap L(c)}.$$

The rest of the proof follows immediately by Theorem 3.8 and the proof of (iv)⇒(i) of this theorem. □

Combining Theorems 4.5 and 4.11 we obtain the following result on the existence of a $\{1,2\}$-inverse belonging to the intersection of an $\mathscr{R}$-class and an $\mathscr{L}$-class. The equivalence (i)⇔(iv) can be seen as the (b,c)-inverse version of [44, Corollary 9].

Theorem 4.12. *Let S be a semigroup and $a,b,c \in S$. Then the following statements are equivalent:*

(i) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$;*
(ii) *there exist a $\{1,2\}$-inverse of a contained in R_b and a $\{1,2\}$-inverse of a contained in L_c;*
(iii) *there exists $u \in S$ such that $b = bucab$, $c = cabuc$ and $a = abuca$;*
(iv) *ca and ab are trace products.*

If all these statements are valid, then the unique $\{1,2\}$-inverse x of a contained in the $\mathscr{H}$-class $R_b \cap L_c$ can be represented by

$$x = yaz = buc, \tag{6}$$

for arbitrary $y \in a\{2\}_{R_b} = a\{1\}_{R(b)}$ and $z \in a\{2\}_{L_c} = a\{1\}_{L(c)}$, and an arbitrary $u \in S$ satisfying $b = bucab$ or $c = cabuc$ or $a = abuca$.

Proof. The implication (i)⇒(ii) is clear.

(ii)⇒(i). Suppose that there exist $y \in a\{1,2\}_{R_b}$ and $z \in a\{1,2\}_{L_c}$, and set $x = yaz$. Then

$$xax = yazayaz = yayaz = yaz = x \qquad \text{and} \qquad axa = ayaza = aza = a,$$

which means that $x \in a\{1,2\}$.

On the other hand, $x = yaz \in R(y) = R(b)$ and $b = yab = yazab = xab \in R(x)$, and hence, $R(x) = R(b)$, i.e., $x \in R_b$. In the same way we obtain that $x \in L_c$. Therefore, we have proved that $x \in a\{1,2\}_{R_b \cap L_c}$.

(i)⇒(iii). Let there exists $x \in a\{1,2\}_{R_b \cap L_c}$. By Theorem 4.5 and Remark 4.4 it is concluded that $x = buc$, for some $u \in S$ such that $b = bucab$ and $c = cabuc$, and since x is a $\{1\}$-inverse of a we conclude $a = axa = abuca$.

(iii)⇒(i). Let there exists $u \in S$ such that $b = bucab$, $c = cabuc$ and $a = abuca$. An application of Theorem 4.5 and Remark 4.4 lead to the conclusion $buc \in a\{2\}_{R_b \cap L_c}$, and by $a = abuca$ it follows that buc is a $\{1\}$-inverse of a.

Further, suppose that the statements (i)–(iii) are true. Let x be the unique $\{1,2\}$-inverse of a contained in $R_b \cap L_c$. The proof of the first equality in (6) is included in the proof of (ii)⇒(i). Consider an arbitrary element $u \in S$ satisfying one of the conditions $b = bucab$, $c = cabuc$ and $a = abuca$. If $b = bucab$, by Theorem 4.5 and Remark 4.4, buc is the unique outer inverse of a contained in $R_b \cap L_c$, whence $x = buc$. In the same way we show that if $c = cabuc$ is satisfied, then $x = buc$. Finally, if $a = abuca$, then $cab = cabucab$, which means that $u \in cab\{1\}$, and Theorem 4.9 initiates $x = buc$.

(ii)⇔(iv). This statement follows immediately from Theorems 3.12 and 3.13. □

Let us give several remarks concerning the previous theorem.

Remark 4.5. (a) As we have seen in Example 4.7, the existence of {2}-inverses of a in R_b and L_c does not guarantee the existence of a {2}-inverse of a in $R_b \cap L_c$, but it is necessary to fulfill some additional requirements that would link two sets of {2}-inverses, such as the requirement $bu = vc$ in (v) of Theorem 4.5 or $b(ab)^{(1)} = (ca)^{(1)}c$ in (viii) of the same theorem. In contrast, the previous theorem shows that the existence of {1,2}-inverses of a in R_b and L_c guarantees the existence of a {1,2}-inverse of a in $R_b \cap L_c$, without any additional requirements.

(b) According to Theorem 4.12, a necessary and sufficient condition for the existence of a {1,2}-inverse of a contained in $R_b \cap L_c$ is solvability of the system of equations $b = bucab$, $c = cabuc$, $a = abuca$, where u is an unknown taking values in S. However, if this system is solvable, then all individual equations have the same set of solutions. Indeed, if $b = bucab$, then buc is a (b,c)-inverse of a, so $c = cabuc$, and by the fact that buc is also an inner inverse of a it follows that $a = abuca$. In the same way we show that $c = cabuc$ implies $b = bucab$ and $a = abuca$. Finally, if $a = abuca$, then $u \in cab\{1\}$ and again we obtain that buc is a (b,c)-inverse of a, whence $b = bucab$ and $c = cabuc$.

(c) The theorem also provides two ways for computing the {1,2}-inverse of a contained in the $\mathscr{H}$-class $R_b \cap L_c$, whenever it exists. The first one is based on solving any of the equations $b = bucab$, $c = cabuc$ and $a = abuca$. In this case, we actually compute the {2}-inverse x of a contained in $R_b \cap L_c$, and if the conditions of existence of a {1,2}-inverse of a contained in $R_b \cap L_c$ are satisfied, then x must also be a {1}-inverse of a.

The second way is to compute any {1,2}-inverse y of a in R_b and any {1,2}-inverse z of a in L_c, and then the {1,2}-inverse x of a contained in $R_b \cap L_c$ is computed as $x = yaz$.

Example 4.13. Consider the set of matrices

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} \,\middle|\, a_1, a_2 \in \mathbb{R}, a_1 \neq 0 \right\}.$$

It is easy to check that S is a subsemigroup of the semigroup $M_\varnothing(\mathbb{R})$ from Example 2.8 in which $\mathscr{R} = \mathscr{D} = S \times S$ and $\mathscr{L} = \mathscr{H}$, and for an arbitrary $C \in S$ the $\mathscr{L}$-class ($\mathscr{H}$-class) of S containing C is given by

$$L_C = H_C = \{\lambda C \,|\, \lambda \in \mathbb{R},\ \lambda \neq 0\}.$$

On the other hand, for an arbitrary $A \in S$, if

$$A = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}, \quad a_1, a_2 \in \mathbb{R}, a_1 \neq 0,$$

then

$$A\{1\} = A\{2\} = A\{1,2\} = \left\{ \begin{bmatrix} 1/a_1 & x_2 \\ 0 & 0 \end{bmatrix} \,\middle|\, x_2 \in \mathbb{R} \right\}.$$

Consequently, for an arbitrary $C \in S$, if

$$C = \begin{bmatrix} c_1 & c_2 \\ 0 & 0 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}, c_1 \neq 0, \qquad X = \frac{1}{a_1 c_1} \cdot C = \begin{bmatrix} 1/a_1 & c_2/a_1c_1 \\ 0 & 0 \end{bmatrix},$$

then X is a $\{1,2\}$-inverse of A contained in the $\mathscr{H}$-class of C. Hence, any matrix from S has a $\{1,2\}$-inverse in each $\mathscr{H}$-class of S.

Let us note that S is a regular subsemigroup of $M_{\varnothing}(\mathbb{R})$, and according to the remark given at the very end of Section 2, Green's relations $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{H}$ on S can be represented by means of ranges and null spaces, as in Example 2.8.

The last result of this section provides existence conditions and characterizations of group inverses through outer and inner inverses belonging to prescribed Green's equivalence classes and solutions of certain linear equations in a semigroup.

Theorem 4.14. *Let S be a semigroup and $a \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{H}$-class H_a;*
(ii) *there exist an outer inverse of a contained in R_a and an outer inverse of a contained in L_a;*
(iii) *there exists an inner inverse of a contained in the quasi-ideal $R(a) \cap L(a)$;*
(iv) *there exist an inner inverse of a contained in $R(a)$ and an inner inverse of a contained in $L(a)$;*
(v) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{R}$-class R_a;*
(vi) *there exists a $\{1,2\}$-inverse of a contained in the $\mathscr{L}$-class L_a;*
(vii) *there exist $u,v \in S$ such that $a = aua^3$ and $a = a^3va$;*
(viii) *there exist $u,v \in S$ such that $a = aua^2$ and $a = a^2va$;*
(ix) *there exists $w \in S$ such that $a = awa^2$ and $a = a^2wa$;*
(x) *there exist $s,t \in S$ such that $a = a^2s$ and $a = ta^2$;*
(xi) *a is group invertible.*

If these statements are true, then

$$a^{\#} = auaua = ava = yaz = tas = t^2a = as^2, \tag{7}$$

for arbitrary $u,v,s,t \in S$ such that $a = aua^2$, $a = ava^3$, $a = a^2s$ and $a = ta^2$, and arbitrary $y \in a\{2\}_{R_a}$ and $z \in a\{2\}_{L_a}$.

Proof. The equivalence of (ii), (iv) and (viii) follows by Theorems 3.10 and 3.11, the equivalence of (iii) and (iv) is obtained from Theorem 4.11 by putting $b = c = a$, and the equivalence of the statements (v)–(ix) is obtained from Theorems 3.12 and 3.13 by putting $b = a$ and $c = a$. The implication (i)⇒(vii) is obtained from Theorem 4.5 by putting $b = c = a$, and the implication (vii)⇒(viii) is obvious.

Further, (x) is equivalent to $a^2 \in H_a$, and according to Green's Theorem (Theorem 2.4), this is equivalent to the claim that H_a is a group. This proves the equivalence of the statements (x) and (xi).

The implication (i)⇒(ii) is clear, and if a is group invertible, then $a^{\#}$ is an outer inverse of a contained in H_a, which means that (xi)⇒(i).

Further, suppose that the claims (i)–(xi) are true. By Theorems 3.12 and 3.13 we obtain $a\{2\}_{R_a} = a\{1,2\}_{R_a}$ and $a\{2\}_{L_a} = a\{1,2\}_{L_a}$. Now, for any $u \in S$ satisfying $a = aua^2$ it follows $au \in a\{2\}_{R_a} = a\{1,2\}_{R_a}$, whence $a = a(au)a = a^2ua$.

If we set $x = auaua$, then it is easy to check that $a = axa$, $x = xax$ and $ax = xa$, which means that $a^{\#} = x = auaua$. The remaining two equalities $a^{\#} = ava$, for $v \in S$ such that $a = ava^3$, and $a^{\#} = yaz$, for $y \in a\{2\}_{R_a} = a\{1,2\}_{R_a}$ and $z \in a\{2\}_{L_a} = a\{1,2\}_{L_a}$, follow directly from Theorem 4.12, by putting $b = c = a$. Finally, let $s, t \in S$ such that $a = a^2 s = ta^2$. It is easy to check that $tas = a^{\#}$ and $ta = as$, whence $a^{\#} = tas = t^2 a = as^2$. □

Remark 4.6. According to the proof of the previous theorem, if both equations $a = aua^2$ and $a = a^2 va$ are solvable, then they have the same set of solutions and to compute the group inverse of a it is enough to solve only one of them. This also holds for equations $a = aua^3$ and $a = a^3 va$.

5. Various extensions of a (b, c)-inverse and an inverse along an element

As we have pointed out earlier, various properties of a (b, c)-inverse and an inverse along an element have been considered in different contexts in a number of recent papers. In addition, various extensions of these concepts have recently emerged, and in this section we will compare them with the concepts we have introduced in this paper.

Bapat et al. [2] dealt with outer inverses $x \in a\{2\}$ which satisfy $xS \subseteq bS$ and $Sx \subseteq Sc$, where S is a semigroup and $a, b, c \in S$. One can easily show that they are outer inverses of a belonging to the quasi-ideal $R(b) \cap L(c)$, so the result from [2] regarding such outer inverses can be restated as follows.

Theorem 5.1. *Let S be a semigroup and $a, b, c \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the quasi-ideal $R(b) \cap L(c)$;*
(ii) *$cab \in S^{(2)}$.*

If these statements are true, then

$$a\{2\}_{R(b)\cap L(c)} = \{b(cab)^{(2)}c \mid (cab)^{(2)} \in cab\{2\}\}. \tag{8}$$

Relaxing the conditions of the above theorem, we provide an existence condition and a characterization of outer inverses that belong to the principal right ideal $R(b)$ generated by a given element $b \in S$.

Theorem 5.2. *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the principal right ideal $R(b)$;*
(ii) *$ab \in S^{(2)}$.*

If these statements are true, then

$$a\{2\}_{R(b)} = \{b(ab)^{(2)} \mid (ab)^{(2)} \in ab\{2\}\}. \tag{9}$$

Proof. (i)⇒(ii). Let $x \in a\{2\}$ and $x \in R(b)$. Then $x = bu$, for some $u \in S^{\mathbb{1}}$, and $y = uabu = uax$ satisfies

$$yaby = uaxabuax = uaxaxax = uax = y.$$

Thus, $ab \in S^{(2)}$.

(ii)⇒(i). Let $y \in ab\{2\}$. Then for $x = by$ we have that $xax = byaby = by = x$ and $x \in R(b)$.

Next, suppose that the statements (i) and (ii) are true. The proof of (ii)⇒(i) implies $\{b(ab)^{(2)} \mid (ab)^{(2)} \in ab\{2\}\} \subseteq a\{2\}_{R(b)}$. On the other hand, by the proof of (i)⇒(ii), the identity $x = xax = buax = by$ is true for every $x \in a\{2\}_{R(b)}$, such that $y \in ab\{2\}$, whence $a\{2\}_{R(b)} \subseteq \{b(ab)^{(2)} \mid (ab)^{(2)} \in ab\{2\}\}$. Thus, (9) holds. □

Due to duality, the following assertion is also true.

Theorem 5.3. *Let S be a semigroup and $a, c \in S$. Then the following statements are equivalent:*

(i) *there exists an outer inverse of a contained in the principal left ideal $L(c)$;*

(ii) $ca \in S^{(2)}$.

If these statements are true, then

$$a\{2\}_{L(c)} = \{(ca)^{(2)}c \mid (ca)^{(2)} \in ca\{2\}\}. \tag{10}$$

Recently, one-sided extensions of a (b, c)-inverse and an inverse along an element emerged. First, Zhu et al. [81] extended the concept of an inverse along an element by splitting the conjunction of Mary's conditions (M1) and (M2), which defines an inverse along an element, into two separate conditions:

(M_ℓ) $x \in L(d)$ and $xad = d$;

(M_r) $x \in R(d)$ and $dax = d$;

where a, d and x are elements of a semigroup S. If there exists $x \in S$ which satisfies (M_ℓ) then a is said to be *left invertible* along d and x is called a *left inverse* of *a along d*. Analogously, if there is $x \in S$ which satisfies (M_r) then a is said to be *right invertible* along d and x is called a *right inverse* of *a along d*.

On the other hand, Drazin [24] (see also Ke et al. [40], Wang and Mosić [68]) extended the concept of a (b, c)-inverse by splitting the conjunction of Drazin's conditions (D1′) and (D2), which defines a (b, c)-inverse, into two separate conditions:

(D_ℓ) $x \in L(c)$ and $xab = b$;

(D_r) $x \in R(b)$ and $cax = c$.

where a, d and x are elements of a semigroup S. If there exists $x \in S$ which satisfies (D_ℓ) then a is said to be *left (b, c)-invertible* and x is called a *left (b, c)-inverse* of a. Similarly, if there is $x \in S$ which satisfies (D_r) then a is said to be *right (b, c)-invertible* and x is called a *right (b, c)-inverse* of a.

Various properties of one-sided invertibility along an element, in the contexts of semigroups and rings, were described by Zhu et al. [81] and Chen et al. [16]. Among other things, Zhu et al. [81] proved that an element a of a semigroup S is left invertible along an element $d \in S$ if and only if $d \in L(dad)$, or equivalently, $d \mathscr{L} dad$, and that a is right invertible along d if and only if $d \in R(dad)$, or equivalently, $d \mathscr{R} dad$. These two results are direct generalizations of Mary's result given here as Corollary 4.6.

In the case when S is an involutive semigroup, Zhu et al. [81] proved that an element a is left invertible along a^* if and only if it is right invertible along a^*, which is also equivalent to the claim that a is MP-invertible. Also, any of these claims is equivalent to the claim that a^* is left invertible (or right invertible) along a.

Basic properties of one-sided (b,c)-inverses were studied by Drazin [25], in the context of semigroups, and by Wang and Mosić [68] and Ke et al. [40], in the context of rings, whereas one-sided (b,c)-inverses of matrices were studied by Benítez et al. [6]. In particular, for a semigroup S and $a,b,c \in S$, Drazin [25] noted that a is left (b,c)-invertible if and only if there is $v \in S$ such that $b = vcab$, which is clearly equivalent to $b \,\mathscr{L}\, cab$ (see also [6,40]). Analogously, a is right (b,c)-invertible if and only if there exists $u \in S$ such that $c = cabu$, which is equivalent to $c \,\mathscr{R}\, cab$. An element a may have multiple left or right (b,c)-inverses, of which not all must be outer inverses of a, but if it has both left and right (b,c)-inverses, then they must be unique and equal to each other, in which case they are a unique (b,c)-inverse of a [25]. Strongly left (b,c)-invertible elements, defined as left (b,c)-invertible elements whose every left (b,c)-inverse is its outer inverse, have been studied by Wang and Mosić [68]. Wang and Mosić [68] and Wang et al. [69] have also studied one-sided extensions of core inverses in the context of rings.

It is interesting to note that in the context of matrices left and right invertibility along an element are mutually equivalent, and therefore, they are equivalent to invertibility along an element, but this does not hold for left and right (b,c)-invertibility (cf. [6]).

When the conjunction of Mary's conditions (M1) and (M2) is splitted into two new conditions, this could also been done in another way, replacing the places of the equations $xad = d$ and $dax = d$. This gives the conditions:

(M'_ℓ) $x \in L(d)$ and $dax = d$;
(M'_r) $x \in R(d)$ and $xad = d$.

Similarly, the conjunction of Drazin's conditions (D1′) and (D2) could been splitted such that the following conditions are obtained:

(D'_ℓ) $x \in L(c)$ and $cax = c$;
(D'_r) $x \in R(b)$ and $xab = b$.

Clearly (M'_ℓ) is essentially the same condition as (D'_ℓ), and (M'_r) is the same as (D'_r). If x satisfies (D'_ℓ), according to Theorem 3.6 it is concluded that x is an outer inverse of a contained in the $\mathscr{L}$-class L_c, and if x satisfies (D'_r), according to Theorem 3.4, x is an outer inverse of a contained in the $\mathscr{R}$-class R_b. This conclusion means that our one-sided extensions of the concepts of a (b,c)-inverse and an inverse along an element, discussed in Theorems 3.4 and 3.6, are different from those proposed by Zhu et al. [81] and Drazin [25], and have the advantage that always are outer inverses of the underlying element and are more precisely localized.

In a series of papers (b,c)-inverses in rings and their extensions have been studied through annihilators. For any $a \in R$, where R is a ring (usually with an identity), the *right annihilator* a° of a and the *left annihilator* $^\circ a$ of a are sets defined as follows:

$$a^\circ = \{x \in R \,|\, ax = 0\}, \quad {}^\circ a = \{x \in R \,|\, xa = 0\}.$$

It is easy to verify that a° is a right ideal and $^\circ a$ is a left ideal of R. Of course, in the context of semigroups, this definition can only be applied to semigroups with zero. However,the methodology common to working with annihilators in rings cannot be applied because it is mainly based on the use of subtraction. A concept of annihilators suitable for use in arbitrary semigroups has been recently proposed by Drazin [27]. For a semigroup S and $a \in S$, the *right annihilator* $\operatorname{rann}(a)$ of a and the *left annihilator* $\operatorname{lann}(a)$ of a are equivalence relations on $S^{\mathbb{1}}$ defined as follows:

$$\operatorname{rann}(a) = \{(r,s) \in S^{\mathbb{1}} \times S^{\mathbb{1}} \mid ar = as\}, \ \ \operatorname{lann}(a) = \{(r,s) \in S^{\mathbb{1}} \times S^{\mathbb{1}} \mid ra = sa\}. \quad (11)$$

Strictly speaking, these concepts are not generalizations of annihilators in rings, but in a sense they are. Namely, in a ring R with identity these concepts are interchangeable due to the fact that for all $a, r, s \in R$ the following is true

$$(r,s) \in \operatorname{rann}(a) \ \Leftrightarrow \ r - s \in a^\circ, \ \ (r,s) \in \operatorname{lann}(a) \ \Leftrightarrow \ r - s \in {}^\circ a.$$

Green's relations were extended in [50,55] to the equivalence relations $\mathscr{L}^\circ$, $\mathscr{R}^\circ$ and $\mathscr{H}^\circ$ as follows: two elements of a semigroup S are related by $\mathscr{L}^\circ$ (resp. $\mathscr{R}^\circ$) in S if and only if they are related by the Green's relation $\mathscr{L}$ (resp. $\mathscr{R}$) over semigroup of S, and $\mathscr{H}^\circ = \mathscr{L}^\circ \cap \mathscr{R}^\circ$.

For $a \in S$, L_a°, R_a° and H_a° denote respectively the $\mathscr{L}^\circ$-, $\mathscr{R}^\circ$- and $\mathscr{H}^\circ$-class of a. It is clear that $\mathscr{L} \subseteq \mathscr{L}^\circ$, $\mathscr{R} \subseteq \mathscr{R}^\circ$ and $\mathscr{H} \subseteq \mathscr{H}^\circ$. As shown in [50] (see also [28,29,42]), the following equivalences are valid for arbitrary $a, b \in S$:

$$a \,\mathscr{L}^\circ\, b \ \Leftrightarrow \ \operatorname{rann}(a) = \operatorname{rann}(b), \quad a \,\mathscr{R}^\circ\, b \ \Leftrightarrow \ \operatorname{lann}(a) = \operatorname{lann}(b). \quad (12)$$

It should be noted that in [50,55] and later papers, these relations were respectively denoted by $\mathscr{L}^*$, $\mathscr{R}^*$ and $\mathscr{H}^*$, but here we use notation with circles because of the shown connection with annihilators.

The relationship between principal left and right ideals and right and left annihilators is shown in the next proposition.

Proposition 5.4. *Let S be a semigroup and $a, b \in S$. Then:*

$$R(a) \subseteq R(b) \ \Rightarrow \ \operatorname{lann}(b) \subseteq \operatorname{lann}(a), \ L(a) \subseteq L(b) \ \Rightarrow \ \operatorname{rann}(b) \subseteq \operatorname{rann}(a). \quad (13)$$

If b is regular, then the reverse implications are also valid, i.e.,

$$R(a) \subseteq R(b) \ \Leftrightarrow \ \operatorname{lann}(b) \subseteq \operatorname{lann}(a), \ L(a) \subseteq L(b) \ \Leftrightarrow \ \operatorname{rann}(b) \subseteq \operatorname{rann}(a). \quad (14)$$

Proof. Implications (13) have been proved by Drazin [27, Proposition 2.4], so we will only prove the reverse implications, assuming that b is regular. Let $\operatorname{lann}(b) \subseteq \operatorname{lann}(a)$ and let $y \in S$ such that $b = byb$. Then $(\mathbb{1}, by) \in \operatorname{lann}(b) \subseteq \operatorname{lann}(a)$, so $a = \mathbb{1}a = bya \in R(b)$, and therefore, $R(a) \subseteq R(b)$. In a similar way we prove that $\operatorname{rann}(b) \subseteq \operatorname{rann}(a)$ implies $L(a) \subseteq L(b)$. □

Drazin [27, Example 3.2] has also provided a counter example which shows that the reverse implications in (13) do not necessarily hold. The semigroup he has constructed is a subsemigroup of the semigroup $\mathbb{F}^{3\times 3}$ of 3×3-matrices over an arbitrary field $\mathbb{F}$, but it is not regular (despite the fact that the semigroup $\mathbb{F}^{3\times 3}$ is regular), so he managed to find a pair of elements (one of which is not regular) such that the reverse implications are not valid. An analogue of the previous proposition for annihilators of rings has been proven in von Neumann's seminal paper on regular rings [64] (see also [75, Lemma 2.3], [38, Lemma 2.8]).

According to Proposition 5.4, Green's relations $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{H}$ coincide respectively with $\mathscr{L}^\circ$, $\mathscr{R}^\circ$ and $\mathscr{H}^\circ$ on all pairs of regular elements. Thus, in any regular semigroup, for example in the semigroup $M_\varnothing(\mathbb{F})$ of matrices over a field $\mathbb{F}$, we have that $\mathscr{L} = \mathscr{L}^\circ$ and $\mathscr{R} = \mathscr{R}^\circ$ and $\mathscr{H} = \mathscr{H}^\circ$.

In the next theorem we consider outer inverses with a prescribed left annihilator, that is, outer inverses belonging to a prescribed $\mathscr{R}^\circ$-class.

Theorem 5.5. *Let S be a semigroup and $a, b \in S$.*
(A) *The following two conditions are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{R}^\circ$-class R_b°;*
(ii) *there exists $x \in S$ such that* $\operatorname{lann}(b) \subseteq \operatorname{lann}(x)$ *and $b = xab$.*

(B) *The condition*

(iii) *$ab \in S^{(1)}$ and $ab \in L_b^\circ$;*

is equivalent to each of the six conditions of Theorem 3.4*, and implies* (i) *and* (ii)*.*
(C) *If $b \in S^{(1)}$, then each of the conditions* (i) *and* (ii) *implies* (iii) *and all the conditions of Theorem* 3.4*.*

Proof. (A) (i)$\Rightarrow$(ii). Let $x \in S$ such that $xax = x$ and $\operatorname{lann}(x) = \operatorname{lann}(b)$. It is clear that $\operatorname{lann}(b) \subseteq \operatorname{lann}(x)$. On the other hand, from $(\mathbb{1}, xa) \in \operatorname{lann}(x) = \operatorname{lann}(b)$ we obtain $b = \mathbb{1}b = xab$.

(ii)$\Rightarrow$(i). From $b = xab$ and $\operatorname{lann}(b) \subseteq \operatorname{lann}(x)$ it follows that $(\mathbb{1}, xa) \in \operatorname{lann}(b) \subseteq \operatorname{lann}(x)$, so $x = \mathbb{1}x = xax$. We also have that $\operatorname{lann}(x) \subseteq \operatorname{lann}(xab) = \operatorname{lann}(b)$, which completes the proof of this implication.

(B) We will prove that (iii) is equivalent to condition (iv) of Theorem 3.4. Note first that $ab \in L_b^\circ$ is equivalent to $\operatorname{rann}(ab) = \operatorname{rann}(b)$, which is further equivalent to $\operatorname{rann}(ab) \subseteq \operatorname{rann}(b)$, because the reverse inclusion is always satisfied.

Let $ab \in S^{(1)}$ and $\operatorname{rann}(ab) \subseteq \operatorname{rann}(b)$. Then $ab = abuab$, for some $u \in S$, so $(\mathbb{1}, uab) \in \operatorname{rann}(ab) \subseteq \operatorname{rann}(b)$, whence $b = b\mathbb{1} = buab$, which was to be proven. Conversely, if $b = buab$ for some $u \in S$, then $ab = abuab$, and hence $ab \in S^{(1)}$, and $\operatorname{rann}(ab) \subseteq \operatorname{rann}(buab) = \operatorname{rann}(b)$.

(C) In accordance with (b), it suffices to show that, assuming that b is regular, (i) implies the condition (i) of Theorem 3.4. Let $x \in a\{2\}$ such that $x \mathrel{\mathscr{R}^\circ} b$. Then both x and b are regular, and according to Proposition 5.4 we obtain that $x \mathrel{\mathscr{R}} b$, which proves the validity of the condition (i) of Theorem 3.4. $\square$

Remark 5.1. Let us note that the above condition (i) is an analogue of condition (i) of Theorem 3.4, whereas the conditions (ii) and (iii) are respectively analogues of conditions (iii) and (v) of the mentioned theorem.

As we have seen, the above condition (iii) is equivalent to all the conditions of Theorem 3.4, but the above conditions (i) and (ii) are weaker than the corresponding conditions of that theorem. In the following example, we will show that these conditions can be strictly weaker than condition (iii), that is, if b is not regular, then (i) and (ii) do not necessarily imply (iii) and the conditions of Theorem 3.4 equivalent to (iii).

Example 5.6. Let S be the multiplicative semigroup of the ring $\mathbb{Z}$ of integers, let $a = 1$ and $b = 2$.

Then $R_b^\circ = \mathbb{Z} \setminus \{0\}$, $R_b = \{2, -2\}$, and $a\{2\} = \{0, 1\}$. Therefore, there exists an outer inverse of a in R_b°, but there is no an outer inverse of a in R_b. We also have that $S^{(1)} = \{0, 1, -1\}$ so $ab = 2 \notin S^{(1)}$, and despite the fact that $ab = 2 \in R_b^\circ = L_b^\circ = \{2, -2\}$ we get that (iii) does not hold.

The next theorem can be proved analogously to the previous one.

Theorem 5.7. *Let S be a semigroup and $a, c \in S$.*
(A) *The following two conditions are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{L}^\circ$-class L_c°;*
(ii) *there exists $x \in S$ such that* $\operatorname{rann}(c) \subseteq \operatorname{rann}(x)$ *and $c = cax$.*

(B) *The condition*

(iii) *$ca \in S^{(1)}$ and $ca \in R_c^\circ$;*

is equivalent to each of the six conditions of Theorem 3.6*, and implies* (i) *and* (ii)*;*
(C) *If $c \in S^{(1)}$, then each of the conditions* (i) *and* (ii) *implies* (iii) *and all the conditions of Theorem* 3.6.

Let S be a semigroup and $a, b, c, x \in S$. Recall that x is a (b, c)-inverse of a if and only if $xax = x$ and $x \in R_b \cap L_c$. In addition to (b, c)-inverses, Drazin [24] introduced several extensions of this notion. By replacing R_b with R_b° and L_c with L_c° he defined the *annihilator (b, c)-inverse* of a, and by replacing only L_c with L_c° (or only R_b with R_b°) he defined the *hybrid (b, c)-inverses* of a. In fact, these definitions were originally given in the context of the rings, but if we use the definition of annihilators given in (11) we get definitions of the corresponding notions in the context of semigroups.

Drazin [24] pointed out that (b, c)-invertibility implies hybrid (b, c)-invertibility, which further implies annihilator (b, c)-invertibility, but the reverse implications do not necessarily hold (Example 5.6 confirms this). Ke et al. [38], Boasso and Kantún-Montiel [9] and Wang et al. [67] proved that the reverse implications hold, for instance, if b and/or c are regular, or cab is regular. Wang et al. [67] also provided new characterizations of (b, c)-invertibility in terms of the properties of the product cab. Hybrid (b, c)-invertibility was also discussed in [75, Lemma 2.13]. Also, it should be noted that the hybrid (b, c)-inverse or annihilator (b, c)-inverse are genuine (b', c')-inverses, for some (any) regular elements $b' \in R_b^\circ$ and $c' \in L_c^\circ$. All the mentioned results have been given in the context of rings, and in the next three theorems we state the corresponding results in the context of semigroups. Although these theorems are proved in an analogous way as the corresponding results in the context of rings, for the sake of completeness we provide proof of the first of these three theorems.

Theorem 5.8. *Let S be a semigroup and $a, b, c \in S$.*
(A) *The following two conditions are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{H}^\circ$-class $R_b^\circ \cap L_c^\circ$;*
(ii) *there exists $x \in S$ such that* $\mathrm{lann}(b) \subseteq \mathrm{lann}(x)$, $\mathrm{rann}(c) \subseteq \mathrm{rann}(x)$, $b = xab$ *and* $c = cax$.

(B) *The condition*

(iii) $cab \in S^{(1)}$ *and* $cab \in L_b^\circ \cap R_c^\circ$;

is equivalent to each of the six conditions of Theorem 4.5, *and implies* (i) *and* (ii).
(C) *If* $b, c \in S^{(1)}$, *then each of the conditions* (i) *and* (ii) *implies* (iii) *and all the conditions of Theorem* 4.5.

Proof. (A) The equivalence of the conditions (i) and (ii) follows directly from Theorems 5.7 and 5.5.

(B) Let the condition (i) of Theorem 4.5 hold, i.e., let there is $x \in S$ such that $xax = x$ and $x \in R_b \cap L_c$. Then x, b, c and cab belong to the same $\mathscr{D}$-class (see Fig. 3), and since $x \in S^{(1)}$ it follows that $b, c, cab \in S^{(1)}$. Now, according to (vi) of Theorem 4.5 and Proposition 5.4, it follows that $cab \in L_b^\circ \cap R_c^\circ$.

Conversely, let (iii) hold. Then $cab = cabwcab$, for some $w \in S$, whence $(\mathbb{1}, wcab) \in \mathrm{rann}(cab) = \mathrm{rann}(b)$ and $(cabw, \mathbb{1}) \in \mathrm{lann}(cab) = \mathrm{rann}(c)$, which yields $b = bwcab$ and $c = cabwc$. This means that the condition (iv) of Theorem 4.5 holds.

Further, since (iii) (or equivalently, (i) of Theorem 4.5) implies the regularity of the elements x, b and c, according to Proposition 5.4 it follows that $R_b = R_b^\circ$ and $L_c = L_c^\circ$. This means that (iii) implies (i).

(C) The proof of this claim follows directly from Proposition 5.4 and the regularity of b and c. □

Theorem 5.9. *Let S be a semigroup and $a, b, c \in S$.*
(A) *The following two conditions are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{R} \cap \mathscr{L}^\circ$-class $R_b \cap L_c^\circ$;*
(ii) *there exists $x \in S$ such that $x \in R(b)$,* $\mathrm{rann}(c) \subseteq \mathrm{rann}(x)$, $b = xab$ *and* $c = cax$.

(B) *The condition*

(iii) $cab \in S^{(1)}$ *and* $cab \in L_b \cap R_c^\circ$;

is equivalent to each of the six conditions of Theorem 4.5, *and implies* (i) *and* (ii).
(C) *If* $c \in S^{(1)}$, *then each of the conditions* (i) *and* (ii) *implies* (iii) *and all the conditions of Theorem* 4.5.

Theorem 5.10. *Let S be a semigroup and $a, b, c \in S$.*
(A) *The following two conditions are equivalent:*

(i) *there exists an outer inverse of a contained in the $\mathscr{R}^\circ \cap \mathscr{L}$-class $R_b^\circ \cap L_c$;*

(ii) *there exists* $x \in S$ *such that* $\mathrm{lann}(b) \subseteq \mathrm{lann}(x)$, $x \in L(c)$, $b = xab$ *and* $c = cax$.

(B) *The condition*

(iii) $cab \in S^{(1)}$ *and* $cab \in L_b^{\circ} \cap R_c$;

is equivalent to each of the six conditions of Theorem 4.5, *and implies* (i) *and* (ii).
(C) *If* $b \in S^{(1)}$, *then each of the conditions* (i) *and* (ii) *implies* (iii) *and all the conditions of Theorem* 4.5.

6. Computational consequences of the presented results

The theoretical results presented so far are very general and imply further theoretical developments, mainly in numerical linear algebra, linear operator theory, matrices with entries in a field, ring or semiring, time-varying matrices, multidimensional arrays, etc. Further, the results stated in the context of semigroups have important computational aspects and implications. In general, the presented results offer a specific and efficient computational-algorithmic framework based on finding solutions to appropriate equations. A selected approach to solving underlying equations can generate the corresponding class of algorithms.

Underlying matrix equations and corresponding generalized inverses are listed in Table 6. In that table, the last column contains references in which the corresponding equation was used. In case the equation has not been used so far, we put "no" in the last cell of the corresponding row.

The approach based on the use of Gradient Neural Networks (GNN) and Zeroing Neural Network (ZNN) is very popular and based on the Frobenius on the error matrix, which is defined based on the matrix equation which is being solved. The GNN evolution design is defined upon the Frobenius norm of the matrix corresponding to the underlying matrix equation. In [61], starting from theoretical characterizations and representations, the GNN evolution was used in solving underlying matrix equations and proposed a number of algorithms for computing outer and inner inverses with predefined range and/or null space for matrices over a field. In the time-varying case, ZNN evolution is defined using the so-called Zhang functions, which represent the underlying equation in the matrix, vector, or scalar case [31,79].

An approach based on finding exact solutions to linear matrix equations underlying the computation of outer inverses was presented in [60]. Corresponding algorithms for symbolic computation of outer generalized inverses of matrices with functional entries were developed.

Also, the hyperpower iterative method with numerous modifications is defined using the powers of the residual matrix corresponding to the underlying matrix equation [62,72].

Theoretical results and algorithms presented in [59] are based on the fact that multidimensional arrays equipped with the Einstein product form a semigroup.

TABLE 1. Matrix equations and the corresponding generalized inverses

No	Source	Equation(s)	Output	References
1	[3,66]	$AX = I, XA = I$	$A^{-1} = X$	[65,78]
2	Theorem 3.4	$BUAB = B$	$A^{(2)}_{\mathscr{R}(B),*} = BU$	[61]
3	Theorem 3.4	$XAB = B, \mathscr{R}(X) \subseteq \mathscr{R}(B)$	$A^{(2)}_{\mathscr{R}(B),*} = X$	[62,72]
4	Theorem 3.6	$CAVC = C$	$A^{(2)}_{*,\mathscr{N}(C)} = VC$	[61]
5	Theorem 3.6	$CAX = C, \mathscr{N}(C) \subseteq \mathscr{N}(X)$	$A^{(2)}_{*,\mathscr{N}(C)} = X$	[62,72]
6	Theorem 3.8	$ABUA = A$	$A^{(1,2)} = BUABU$	no
7	Theorem 3.9	$AVCA = A$	$A^{(1,2)} = VCAVC$	no
8	Theorem 3.10	$AUA^2 = A$	$A^{(2)}_{\mathscr{R}(A),*} = AU$	no
9	Theorem 3.10	$XA^2 = A, \mathscr{R}(X) \subseteq \mathscr{R}(A)$	$A^{(2)}_{\mathscr{R}(A),*} = X$	no
10	Theorem 3.11	$A^2UA = A$	$A^{(2)}_{*,\mathscr{N}(A)} = UA$	no
11	Theorem 3.11	$A^2X = A, \mathscr{N}(A) \subseteq \mathscr{N}(X)$	$A^{(2)}_{*,\mathscr{N}(A)} = X$	no
12	Theorem 4.5	$VCAB = B, CABU = C$	$A^{(2)}_{\mathscr{R}(B),\mathscr{N}(C)} = VC = BU$	[61]
13	Theorem 4.5	$BUCAB = B, CABVC = C$	$A^{(2)}_{\mathscr{R}(B),\mathscr{N}(C)} = BUC = BVC$	[60,61]
14	Theorem 4.5	$BWCAB = B, CABWC = C$	$A^{(2)}_{\mathscr{R}(B),\mathscr{N}(C)} = BWC$	[61]
15	Theorem 4.5	$BUAB = B, CAVC = C,$ $BU = VC$	$A^{(2)}_{\mathscr{R}(B),\mathscr{N}(C)} = BU = VC$	[61]
16	Theorem 4.12	$BUCAB = B, CABUC = C,$ $ABUCA = A$	$A^{(1,2)}_{\mathscr{R}(B),\mathscr{N}(C)} = BUC$	no
17	Theorem 4.14	$AUA^2 = A, A^2VA = A$	$A^\# = AUAUA = AVAVA$	no
18	Theorem 4.14	$AWA^2 = A, A^2WA = A$	$A^\# = AWAWA$	no
19	Theorem 4.14	$AUA^3 = A, A^3VA = A$	$A^\# = AUA = AVA$	no
20	Theorem 4.14	$UA^2 = A, A^2V = A$	$A^\# = UAV = U^2A = AV^2$	no

7. Conclusion and visions of future research

The presented theoretical results provide equivalent conditions for the existence and corresponding characterizations for outer inverses. The results are stated in a very general form, as outer inverses in semigroups belonging to the prescribed Green's $\mathscr{R}$-, $\mathscr{L}$- and $\mathscr{H}$-classes. They are applicable in many mathematical structures due to very general theoretical results. Particularly, the obtained results generalize the well-known and frequently investigated problem of finding outer matrix inverse with the prescribed image or/and kernel in various spaces. Primarily, the corresponding investigations in numerical linear algebra and on linear operators can be expected. Further, the corresponding theoretical research and computational procedures are expectable for matrices with entries over a field, ring or semiring, time-varying matrices, multidimensional arrays, etc. So far, in [59], the authors applied this global algorithmic framework in the tensor case.

In our further research we will introduce the concept of trace factorization, which can be viewed as a semigroup-theoretical generalization of the full-rank factorization of matrices. Some additional characterizations of the group and (b, c)-invertibility in semigroups will be given using the trace factorization. Some new existence criteria, characterizations and representations for $\{1, 3\}$- and $\{1, 4\}$-inverses, MP-inverses, and (dual)core inverses will also be established.

Derived representations of $\{1\}$,$\{2\}$,$\{1, 2\}$,$\{1, 3\}$- and $\{1, 4\}$-inverses, MP-inverses, and (dual)core inverses are based on simple and efficient algorithmic framework which can be described in two global steps:

1. Solve appropriate equation(s);
2. Multiply the solution obtained in the first step by the appropriate elements, if necessary.

The underlying equations can be solved using various methods. The GNN approach was used in [61]. Of course, other computational techniques can be applied in solving required equations, leading to a number of various computational algorithms.

Moreover, there is a number of representations of various classes of generalized inverses which have not been investigated so far. These cases are marked by "no" in the last column of Table 6. This means that all equations corresponding to "no" define a new approach in computation of new classes of generalized inverses, which can be used in further research.

We emphasize that there is another way to turn the set of all matrices over a field (of arbitrary types) into a full semigroup, while preserving the classical matrix product. This can be done using the so-called *semi-tensor product*, an associative product that is defined for matrices of arbitrary types and coincides with the classical matrix product whenever it is defined (cf. [17]). However, such a semigroup will be the subject of our research that will be conducted in the future.

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